Lecture Notes Math 1170 and Math 1175 - Calculus I & II

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Chapter 1

Introduction

Calculus is the study of continuous and smooth variation of quantities related by well defined correspondences. This entails the study of infinitesimal/local variations as well as long-distance/non-local variations that regard infinitely large sets. Two examples are the notion of derivative, which concerns the instantaneous variation of a function, and indefinite integrals, that evaluate the area under a curve that extends from $-\infty$ to ∞. Examples of applications of Calculus outside of pure math can be found virtually in any branch of science and even outside of STEM courses.

Calculus can be traced back to the work of Newton and Leibniz in the theory of differentials, whose main application at the time was the study of celestial mechanics, i.e. the study of the motion of planets. In fact, Newton's laws in physics (mechanics to be precise) are formulated through notions that are studied in calculus. Further applications can be found for instance in biology, where the mathematical models describing population growth use the notion of differentials and are studied through methods due to calculus. In chemistry the study of reactions is done through differential calculus as well. In mechanical engineering as well as electrical engineering, calculus is used to design mechanical/electrical components in an optimal way. More recently, artificial intelligence and machine learning have used optimization techniques from calculus to have computers perform specific tasks. Computational medicine and bioinformatics, as field of applications where artificial intelligence and machine learning are used on a daily basis strongly depend as well on calculus.

These notes are based on the standard textbook [4] used for this course at Idaho State University. More advanced topics can be found in textbooks for Mathematical Analysis such as [1–3].

Chapter 2

Functions

A function is as a law that to any given element from a set (called domain), associates a single element of a target set (called codomain). This means that a function consists of three bits of information. Firstly, one needs to specify a domain (a set). Secondly, one needs to give a codomain (another set). Lastly, a specific procedure that allows to pass from domain to codomain in a unique way is needed. This procedure in practice consists of an equation. To indicate a function one uses the symbol $f : X \longrightarrow Y$, where X and Y are domain and codomain (respectively), and f is the law that allows to pass from X to Y. When X and Y are clear, one can omit them and simply indicate a function as f, but it is important to recall that X and Y are implicitly considered. One can imagine a function as a machine that takes an input and produces an output.

Example 2.0.1. Let $X = \mathbb{R}$, $Y = \mathbb{R}$ and define $f(x) = x^2$. This means that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ takes elements from $\mathbb{R}$, maps them into $\mathbb{R}$, and it does so by taking a number x and squaring it, i.e. producing x^2. We can also consider the function $X = \mathbb{R}_0^+$, $Y = \mathbb{R}$ and define $f(x) = x^2$, where now the domain consists of all nonnegative numbers, rather than all numbers. While the two functions perform the same procedure, they are different because their domains are different.

One common way of depicting (visualizing) functions is through their graph. The graph of $f : X \longrightarrow Y$ is the set of all pairs (x, y) where x is in X and y is in Y, such that $f(x) = y$. The function $f(x) = x^2$ is shown in Figure 2.1.

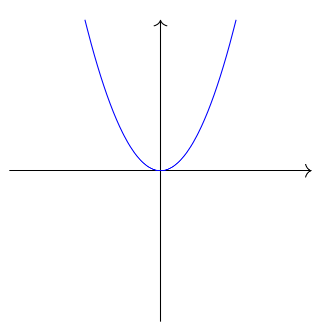


Figure 2.1: Plot of function $f(x) = x^2$. The horizontal axis represents x while the vertical axis is y.

The *vertical line test* is a useful and intuitive way of determining whether a curve on the plane corresponds to the graph of a function. Observe that since we need to have a unique correspondence from domain to codomain, this means that for each element on the horizontal line, there is going to be only on element along the vertical line passing through it. So, if the vertical line passing through x intersects a curve more than once, it means that the graphs does not correspond to any function!

An immportant notion regarding functions is the fact that they tend to increase or descrease.

Definition 2.0.2. A function is said to be increasing if whenever $x_1 < x_2$ we also have $f(x_1) < f(x_2)$. A function is said to be decreasing if whenever $x_1 < x_2$ we have $f(x_1) > f(x_2)$.

A function can have an increasing behavior in cetrain parts of the domain, and a descreasing behavior in other parts. Consider for instance the function $f(x) = x^2$. Then this is decreasing in $(-\infty, 0]$ and it is increasing in $[0, \infty)$. Look at the graph in Figure 2.1!

Operations on functions

There are a number of elementary operations that we can perform on functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, which are inhertied from the operations of $\mathbb{R}$.

- Given two functions f and g, we can define their sum $f+g$ by the rule $(f+g)(x) = f(x)+g(x)$.
- We can define the difference as $(f - g)(x) = f(x) - g(x)$.
- Product $(f \cdot g)(x) = f(x)g(x)$.
- Quotient $(f/g)(x) = \frac{f(x)}{g(x)}$; here we need to be careful to consider the quotient only when $g(x) \neq 0$ for all x.

Another fundamental operation that can be performed on functions is their composition. This operation does not depend on the codomain and domain being $\mathbb{R}$. More specifically, if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two functions, we can define their composition (indicated by $g \circ f$) by the rule

$$g \circ f(x) = g(f(x)).$$

From this definition is apparent that we need the domain of g to be the same as the domain of f, because we would not otherwise be able to apply g on the outpiut of f. In arrow notation, the composition looks like

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

Observe that writing $g \circ f$ has the order of f and g switched! In other words, we apply f first, but we write it on the right. Be careful about this notational fact.

Inverse of a function

It can happen that a function satisfies the property that for any choice of y in Y, there exists a unique element x in X such that $f(x) = y$. The graph of such a function satisfies also the *horizontal* *test*, which is the same as the vertical test, but where the roles of x and y are exchanged. In this case, we can define the inverse function, denoted by the symbol $f^{-1} : Y \longrightarrow X$ which satisfies the following properties: $f^{-1} \circ f(x) = x$ and $f \circ f^{-1}(y) = y$.

The rationale for this definition is the following. If whenever I pick y in Y, there is a unique x in X such that $f(x) = y$, then I can also define $f^{-1}(y) = x$, which is simply the function undoing f. But if I undo f, then I am simply not performing anything on x. So, applying f and f^{-1} gives simply the function that takes x and returns x (quite a boring function). I could also start with f^{-1}, and do the same reasoning with the roles of f and f^{-1} inverted.

Observe that if f^{-1} is the inverse of f, then f is the inverse of f^{-1}, i.e. $(f^{-1})^{-1} = f$.

Chapter 3

Limits and derivatives

The notions of limit and derivative formalize the intuitive concept of infinitesimals. A typical problem where this arises is found in mechanics (part of physics), where we want to compute the velocity of an object. Velocity, is defined as the ratio of space and time, and it represents the space that an object (e.g. a car) travels within a time frame. This is the reason why odomoters in cars report a quantity in miles per hour (i.e. space per time). This is a ratio of space (miles) over time (hours), and it is indicated as

$$v = \frac{\Delta s}{\Delta t},$$

where s indicates space, and t indicates time. However, this quantity does not refer to a precise instant, but rather to a whole time interval. What if we wanted to know the *instantaneous* velocity? This is the velocity at a certain precise time, and not at an interval of time. This would mean that Δt needs to become smaller and smaller. It needs to be infinitesimal! The issue we are facing here is to be able to make sense of this "infinitesimal" meaning. The resulting object, denoted as $\frac{ds}{dt}$ is the notion of derivative, and it is defined through a limit.

This problem relates to the notion of tangent to a curve. Suppose we have two points P and Q over the graph of a function (which is a curve in the plane). Then, consider the line through the points P and Q. See Figure 3.1.

Now, if we let P move to Q and eventually let it overlap on Q, it follows that the line through the two points will touch the graph only at the point $P \equiv Q$. This means that we have found the tangent. Since the slope of the line through P and Q can be written as $\frac{y_P - y_Q}{x_P - x_Q} = \frac{\Delta y}{\Delta x}$, we find that in this case the slope of the tangent is the "limit" $\frac{dy}{dx}$ similarly to the case of instantaneous velocity.

We introduce now the notion of limit. We first do so through a wordy definition, and then give a more symbolic definition that simplty restates the first one.

Definition 3.0.1. Let $f(x)$ be a function and let a be an element of $\mathbb{R}$ such that there exists an interval (b, c) around a which is contained in the domain of f. Then, we say that the limit of f at a is L, and we write

$$\lim_{x \to a} f(x) = L,$$

if we can make $f(x)$ arbitrarily close to L upon taking x sufficiently close to a.

Two important things about the definition are the following. First, a itself does not necessarily need to be in the domain of f, but only an interval around it needs to be. Second, the intuitive

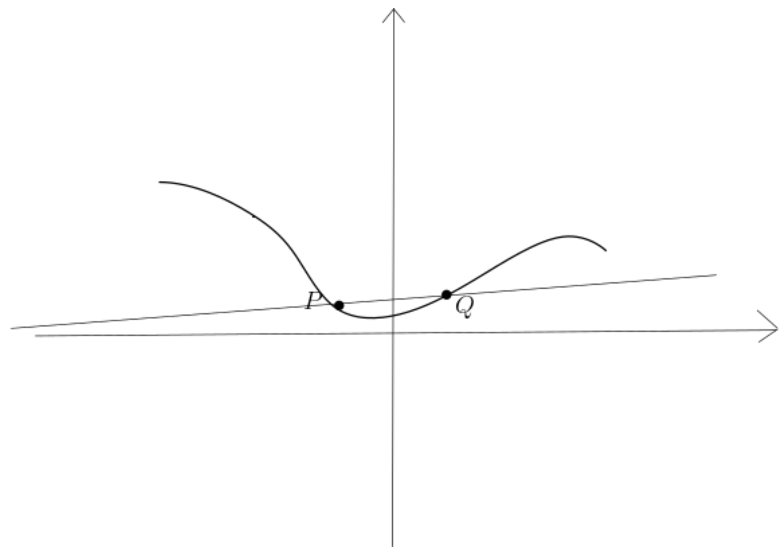


Figure 3.1: Two points on the graph of a function and a line through them

meaning of the definition is that as x gets close to a, then $f(x)$ gets close to L. Here f does not need to reach the value L, though. But rather, we can make the difference $|f(x)-L|$ as small as we want, upon choosing x close to a. We also write sometimes that $f(x) \longrightarrow L$ as $x \longrightarrow a$, to denote limits.

A more formal definition of limit is the following.

Definition 3.0.2. For every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in (a-\delta, a+\delta)$ and $x \neq a$ we have $|f(x) - L| < \epsilon$.

There are some rules that are very helpful in computing limits. In particular, computing limits of cetrain types of functions is relatively easy, as we can apply the substitution principle as described below.

Here we assume that c is a constant, and that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then:

- $\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} f(x)$.

- $\lim_{x\to a} f(x) - g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} f(x)$.

- $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$.

- $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$.

- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} f(x)}$ under the extra assumption that $\lim_{x\to a} g(x) \neq 0$.

For polynomial functions, trigonometric functions, roots, exponentials, logarithms, and rational functions, we further have that calculating limits as x goes to a point a of their domain is simply computed by substitution of a instead of x.

Example 3.0.3. Let us consider the function $f(x) = \frac{x^2-1}{x+1}$. We want to compute the limit of f as x goes to 1. We have that 1 is in the domain of f, and therefore, to compute $\lim_{x\to 1} f(x)$ we just need to substitute 1 in place of x. This means

$$\lim_{x\to 1} \frac{x^2-1}{x+1} = \frac{1^2-1}{1+1} = 0/2 = 0.$$

However, for the function $f(x) = \frac{x^2-1}{x-1}$ we cannot use the same approach, as $x = 1$ is a root of the denominator, and we would be dividing by zero. We can however factorize the numerator and simplify as

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1} x+1.$$

Now we have the limit of the function $g(x) = x + 1$, which we can compute following the rules above (this is polynomial), and we get

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 1+1 = 2.$$

The principle is that whenever substituting makes sense (i.e. no trivial denominators and similar things) we can simply compute the limit by substitution. When this is not possible, we have to try to reduce the problem to an equivalent formulation that allows us to use the substitution.

There are cases where as x approaches some number say a, the function does no stabilize towards any number L, but instead keeps increasing (or decreasing) wihtout ever stopping. One example of such behavior is quite well known to all of us, and it is the function $f(x) = \frac{1}{|x|}$ when we take x closer and closer to zero. Observe that zero is not in the domain of f here, but nonetheless, we can compute the limit, as this is not a requirement of the definition above. There is no L such that f gets closer and closer to it as x gets closer and closer to 0. This happens because f increases without any bound, so it passes the value of any L you fix. In this case, we say that f has a *vertical asymptote* at $x = 0$, and we write

$$\lim_{x\to 0} f(x) = \infty.$$

More generally, we have the following.

Definition 3.0.4. If when x gets closer and closer to a, the function $f(x)$ becomes larger and larger without any bounds, then we write

$$\lim_{x\to a} f(x) = \infty,$$

and say that f has a vertical asymptote. Similarly, if f decreases without bounds and becomes more and more negative, we write

$$\lim_{x\to a} f(x) = -\infty.$$

In this situation we say that f has a vertical asymptote as well.

The same rules for computing the limit given above hold when the limits are infinite, with some careful considerations.

- First, if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = L$ (where L is finite), then we have
$$\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = \lim_{x\to a} f(x) = \pm\infty.$$
Same thing holds exchanging the roles of f and g.

- If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = L \neq 0$, then we have that $\lim_{x\to a} f(x)g(x) = \pm\infty$ where the sign is determined according to the rules. If $\lim_{x\to a} f(x) = \infty$ and $L > 0$ this is $+$, if $\lim_{x\to a} f(x) = \infty$ and $L < 0$ it is $-$. If $\lim_{x\to a} f(x) = -\infty$ and $L > 0$ this is $-$, if $\lim_{x\to a} f(x) = -\infty$ and $L < 0$ it is $+$.

  Warning: If the limit of g is 0, there is not much you can say right away, and this needs to be treated on a singular basis, as we will see in the examples.

  Similar results hold when the role of f and g is exchanged.

- If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = L \neq 0$. Then $\lim_{x\to a} f(x)/g(x) = \pm infty$. The case $\lim_{x\to a} g(x) = 0$ can have some sign oscillations so it needs more care. Also, we have that $\lim_{x\to a} g(x)/f(x) = 0$ (here $\lim_{x\to a} g(x) = 0$ does not cause any problems).

Let us consider now several examples to show the whole discussion up to now.

Example 3.0.5. Find the limit
$$\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2}.$$
Of course, this limit cannot be evaluated directly by substitution, because the denominator becomes zero, and the numerator is zero as well. This is called an *indeterminate form* of type $\frac{0}{0}$.

To solve the issue, multiply both numerator and denominator by $\sqrt{x^2+9}+3$. We get
$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2} &= \lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2}\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}\\ &= \lim_{x\to 0} \frac{x^2+9-9}{x^2(\sqrt{x^2+9}+3)}\\ &= \lim_{x\to 0} \frac{1}{\sqrt{x^2+9}+3}.\end{aligned}$$
Now, we can compute this limit by substitution and we get
$$\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2} = 1/6$$

Example 3.0.6. Compute the limit as $x \longrightarrow 3$ of the function $f(x) = \frac{7x-3}{x^2-9}$. Observe that the denominator is problematic in the sense that it is a root of the polynomial at the denominator. However, as x goes to 3, the numerator goes to $21 - 3 = 18$. This means that the denominator goes to zero, and the numerator is a finite number. In both cases, the sign is $+$ around 3 (for the denominator this is due to the presence of the square).

This means that we divide a number around 18 by smaller and smaller (positive) numbers. This is the same situation as we have for the function $f(x) = \frac{1}{|x|}$. The limit is therefore ∞. This situation is written intutitvely as $\frac{1}{0}$ (this is not an actual division, but rather a way of easily indicating what is going on).

When defining the limit of a function, we have considered intervals around a point a where points were coming both from the left and the right of it. However, we can consider only limits where we consider points only on the right, or only from the left. These limits are indicated by the symbols $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$, respectively.

We now consider the definition and evaluation of limit when x grows or decreases without any bound. These limits consider the behavior of the function when $x \longrightarrow \pm\infty$, rather than when it goes to a specific numerical value L. If f becomes arbitrarily close to a specific value as x keeps increasing or decreasing, we say that this specific numerical value is its limit as x goes to ∞ or $-\infty$. We indicate such limits by the symbol

$$\lim_{x\longrightarrow\pm\infty} f(x) = L,$$

and in such situation we say that L is right/left *horizontal asymptote*.

It might also happen that as x goes to $\pm\infty$, the function f incfreases or decreases unboundedly. In such situation we write

$$\lim_{x\longrightarrow\pm\infty} f(x) = \pm\infty.$$

Example 3.0.7. Consider the function $f(x) = \frac{x^3-3x^2+2}{x^2+3x-7}$. Then, we can write

$$f(x) = \frac{x^3(1-3/x+2/x^3)}{x^2(1+3/x-7/x^2)} = x\frac{1-3/x+2/x^3}{1+3/x-7/x^2}.$$

As x goes to ∞, the terms $3/x, 2/x^3, 7/x^2$ all go to zero, so that the fraction tends to 1, but the multiplying factor of x gives a limit of ∞.

3.1 The Squeeze Theorem

Theorem 3.1.1. *If $f(x) \leq g(x)$ for all x near to a, then we have that*

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

In particular, if $f(x) \leq g(x) \leq h(x)$ for all x near a, and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \lim_{x\to a} h(x).$$

The Squeeze Theorem says that if we can find (at least around a) two functions having the same limit, and bounding $g(x)$ from above and below, then the limit of $g(x)$ has to be the same as the limit of these two functions. The reason why this happens is that once $g(x)$ is trapped between two functions going to the same place from two different sides (above and below), then $g(x)$ cannot go anywhere but where $f(x)$ and $h(x)$ go. In other words, $g(x)$ is squeezed to the same limit as well.

Example 3.1.2. Consider the function $f(x) = x^2\sin(x)$. We know that $|\sin(x)| \leq 1$, so $\sin(x)$ is always between -1 and 1. This means that we can bound $f(x)$ as

$$-x^2 \leq x^2\sin(x) \leq x^2.$$

Since both $-x^2$ and x^2 go to zero, as x goes to 0, we find that

$$\lim_{x\to 0} f(x) = 0$$

as well, applying the Squeeze Theorem.

3.2 Continuous functions

A function can satisfy certain regularity properties that make it simpler to deal with. In fact, one of these properties was encountered in the computations of limits, where we have seen that some functions are simple enough that to compute their limit to a point lying in its domain we can simply plug in the value in the function. Such functions are said to be continuous. More specifically, we have

Definition 3.2.1. A function f is said to be continuous at the number a in its domain, if

$$\lim_{x \to a} f(x) = f(a).$$

Observe that if the limit as x goes to a of f does not exist, then f is discontinuous at a by definition. So, there are two different ways a function can be discontinuous at a point a of its domain. It can fail to have a limit at a, or it can have a limit which is not equal to $f(x)$.

In the second case, when $\lim_{x \to a} f(x)$ exists but it is different from $f(a)$, we can make f continuous at a by redefining f at a as $f(a) = \lim_{x \to a} f(x)$. This basically fixes the issue and makes it continuous at a. In the first case, i.e. when f does not have a limit, there is not much we can do.

Definition 3.2.2. A function f is said to be continuous from the right at the number a in its domain, if

$$\lim_{x \to a^+} f(x) = f(a).$$

A similar definition holds for limits on the left.

Definition 3.2.3. A function f is said to be continuous on the subset Z of its domain X if f is continuous at each point of Z. If f is continuous over all the points of its domain, then f is simply said to be *continuous*.

Putting together continuous functions produces continuous functions still.

Theorem 3.2.4. *Let f and g be continuous functions. Then, the following hold*

- *$f + g$, $f - g$ and cf are continuous (for any choice of a number c).*
- *fg is continuous.*
- *If g is not zero on its domain, f/g is continuous.*
- *If f and g are such that their composition makes sense, then $g \circ f$ is continuous.*

Proposition 3.2.5. *Polynomial functions, rational functions, root functions, trigoinometric functions, exponential functions and logarithmic functions are all continuous over their domains.*

Remark 3.2.6. Observe that the domain of polynomial and exponential functions is $(-\infty, \infty)$. Rational functions might have "punctures" where the denominator is zero, and they are therefore not defined. Root functions might only be defined on the non-negative numbers, depending on the root. logarithmic functions are defined on the (strictly) positive numbers.

3.3 Intermediate Value Theorem (IVT)

We explore now a very useful property of continuous functions, which is stated in the Intermediate Value Theorem (IVT).

Theorem 3.3.1. *Suppose that f is a continuous function over the interval $[a,b]$, and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, there exists a number c with $a < c < b$ such that $f(c) = N$.*

Remark 3.3.2. The meaning of the IVT is that continuous functions do not have jumps over intervals. In fact, the theorem is saying that between $f(a)$ and $f(b)$, the function f attains every point. In other words, it does not make a jumpt.

Example 3.3.3. A fundamental thing in the theorem is that f is continuous over the interval. So, if f is not continuous, or f is continuous but we are not considering an interval, the result of the IVT might not be true. For instance, consider the function defined by the law

$$f(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq 1 \\ 1 & \text{if} \quad 2 \leq x \leq 3 \end{cases}$$

Then f is continuous over its domain, but between $f(0) = 0$ and $f(3) = 1$ there is a gap. This is due to the fact that we are not considering a single interval, but the union of two intervals.

Consider now the funtion $f : [0,2] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq 1 \\ 1 & \text{if} \quad 1 < x \leq 2 \end{cases}$$

This function too does not attain all elements between $f(0)$ and $f(2)$, and we are now considering the interval $[0,2]$. What went wrong this time is that f is not continuous.

Problem 3.3.4. Show what went wrong in the previous example by drawing the graph of the functions.

A very useful application of the IVT in practice is to show that there exist solutions to certain equations. Moreover, the IVT would also show an interval in which such solutions lie, therefore giving a way of roughly approximating the solutions.

Example 3.3.5. Consider the equation $4x^3 - 6x^2 + 3x - 2 = 0$, and show that a solution to it exists. Estimate it.

Consider the function $f(x) = 4x^3 - 6x^2 + 3x - 2$ which is continuous because it is a polynomial. Consider $x = 0$ and evaluate f at it: $f(0) = -2$. Consider $f(2) = 12$. Then this means (by the IVT) that $f(x)$ attains all values between -2 and 12. But such values also include 0, which means that $f(x) = 0$ for some point between $x = 0$ and $x = 2$. We can actually see that $f(1) = -1$, so that we can repeat the previous reasoning with $x = 1$ and $x = 2$ and say that $f(x) = 0$ has a solution in $(1,2)$. In fact, we can repeat this approach, and see that $f(1.5) > 0$. So, a solution exists between $x = 1$ and $x = 1.5$. So proceeding, we can improve our guess for a solution of $f(x) = 0$ by taking half of the interval, evaluating whether f is positive or negative at that value, and selecting the interval that has extremes with opposite signs. We get better approximations each time that we perform this procedure.

3.4 Derivatives

We have seen, at the beginning of the section, that in order to compute the tangent to a function, we need to get the two points lying on the graph of a function closer and closer. The notion of closer and closer is formalized through the concept of limit, which we have extensively considered in the previous part of the section.

We can state this definition as follows.

Definition 3.4.1. The slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is given (if it exists) by the value

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Also, we have seen that the velocity of a moving object was obtained as a fraction which we wanted to evaluate with smaller and smaller time lapses. This is as well a limit, and we have

Definition 3.4.2. The instantaneous velocity v of a moving object is obtained as the limit (if it exists) of the ratio

$$v = \lim_{\Delta t \longrightarrow 0} \frac{\Delta s}{\Delta t}.$$

Both limits are a special case of the more general notion of *derivative*.

Definition 3.4.3. The derivative of a function f at a, where a is in its domain, is given by the limit (if it exists)

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Equivalently (why?) this can be written as

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 3.4.4. Let us now use the definition to compute the derivative of $f(x) = x^2 - 3x$ at the point $x = 1$. Using the the second fraction in the definition we have

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{(1+h)^2 - 3(1+h) - (1-3)}{h} \\ &= \lim_{h\to 0} \frac{1 + h^2 + 2h - 3 - 3h + 2}{h} \\ &= \lim_{h\to 0} \frac{h^2 - h}{h} \\ &= \lim_{h\to 0} h - 1 = -1 \end{aligned}$$

We can try to obtain the derivative of a function on any point of its domain. If the derivative exists, then we have a way of associating to this value a another value defined as $f'(a)$. This means that the derivative is itself a function defined over the set of points such that $f'(a)$ exists finite (as a limit). This function is indicated by $f'(x)$ and it simply indicates the derivative at the point a, as we let a vary.

The domain of f' is the set of numbers where $f'(a)$ exists, and this does not need to be the same as the domain of f. It is in general at most the same as the domain of f (in case all points are such that f' exists!).

Example 3.4.5. Consider the function $f(x) = \sqrt{x}$. We want to compute its derivative using the definition, and we want to determine what the domain of f' is. At a generic point x, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\ &= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

From the computation we also see that $x = 0$ is a point where the limit does not exist, and therefore it is not in the domain of $f'(x)$. Therefore, we get $f'(x) = \frac{1}{2\sqrt{x}}$ with its domain being $(0, \infty)$.

Other notations to indicate the derivative function are $\frac{df}{dx}$, $\frac{d}{dx}f(x)$ or $D_x f(x)$.

There are cases where the derivative function is not defined, but the limit of Definition 3.4.3 is not infinite (as in the previous example), but it does not exists still.

Example 3.4.6. Consider the function $f(x) = |x|$. To compute the derivative we have

$$f'(x) = \lim_{h\to 0} \frac{|x+h|-|x|}{h}.$$

Now, recall that $|x| = x$ whenever $x \geq 0$, and $|x| = -x$ whenever $x < 0$. So, we have to distinguish the two cases where $x \geq 0$ and where $x < 0$. For > 0 we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{|x+h|-|x|}{h} \\ &= \lim_{h\to 0} \frac{x+h-x}{h} \\ &= \lim_{h\to 0} 1 = 1, \end{aligned}$$

while for $x < 0$ we get

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{|x+h|-|x|}{h} \\ &= \lim_{h\to 0} \frac{-x-h+x}{h} \\ &= \lim_{h\to 0} -1 = -1, \end{aligned}$$

This means that the right and left limits at zero do not coincide, which also means that the limit does not exist at zero. This implies that $f(x)$ is not differentiable at $x = 0$. However, for $x < 0$ and $x > 0$ the right and left limits are the same and they exist (either -1 or 1, repsectively). This means that $f(x)$ is differentiable everywhere but at $x = 0$.

We want to show now that differentiability is a stronger condition than continuity. By this we mean that a continuouus function is also continuous, although the opposite is not always true. In other words, there exist functions that are continuous but not differentiable (two examples where seen before!).

Theorem 3.4.7. *If f is a differentiable function at $x = a$, then it is continuous at $x = a$ as well.*

Proof. We want to show that $\lim_{x\to a} f(x) - f(a) = 0$, which would imply that $f(x)$ goes to $f(a)$ when x goes to a. We have

$$\begin{aligned}\lim_{x\to a} f(x) - f(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} x - a \\ &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x - a) \\ &= f'(a) \cdot 0 = 0.\end{aligned}$$

This shows that $f(x) \longrightarrow f(a)$ as x goes to a, which means that f is continuous at a. □

Chapter 4

Differentiation rules

We now obtain several useful rules of differentiation which allow us to compute derivatives directly from a given function, without needing to use the definition each time that we want to obtain a derivative.

4.1 Linear combinations and powers

We have the following useful resutl.

Proposition 4.1.1. *Let f and g be differentiable functions, and let a and b be numbers. Then, the function $af + bg$ is differentiable, and its derivative is given by*

$$\frac{d}{dx}[af(x) + bg(x)] = a\frac{df(x)}{dx} + b\frac{dg(x)}{dx}.$$

In other words, we compute the derivatives separately, and then combine them linearly.

Proposition 4.1.2. (Power Law) *Let $f(x) = x^n$ for some real number n. Then, $f(x)$ is differentiable, and it has derivative*

$$f'(x) = nx^{n-1}.$$

Proof. We prove the result only for the simpler case where n is a positive integer (like $n = 0, 1, 2, \ldots$ etc). For a generic number a, we want to compute the limit

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x^n - a^n}{x - a}.$$

Now, since $x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$, we can rewrite our limit as

$$\begin{aligned}
\lim_{x\to a} \frac{x^n - a^n}{x - a} &= \lim_{x\to a} \frac{(x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x - a} \\
&= \lim_{x\to a} x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}.
\end{aligned}$$

To compute the latter we can now simply insert a instead of x, obtaining

$$\begin{aligned}
\lim_{x\to a} \frac{x^n - a^n}{x - a} &= \lim_{x\to a} x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1} \\
&= a^{n-1} + a^{n-1} + \cdots + a^{n-1} + a^{n-1} \\
&= na^{n-1}.
\end{aligned}$$

□

4.2 Application: Derivatives of polynomials

We now use the result above to obtain a very simple way of computing the derivative of polynomials. A polynomial is a function of type $f(x) = a_n x^n + a_{n-1}x^{n-2} + \cdots + a_1 x + a_0$. So, in other words, a polynomial is a function obtained by a linear combination of power functions. In fact, $a_n, \ldots, a_0$ are all numbers, and each term $x^n, \ldots, x, 1$ are all power functions. So, using the results above, to compute the derivative of $f(x)$ we can compute the derivatives of all the powers separately, and then combine them together. Since the derivative of a power is simple to compute, we get a general rule for the differentiation of a polynomial. With $f(x) = a_n x^n + a_{n-1}x^{n-2} + \cdots + a_1 x + a_0$ we have

$$f'(x) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

Observe that all powers dicrease, when taking a derivative, and the last term a_0 disappears.

Example 4.2.1. Compute the derivative of $f(x) = 2x^3 - x^2 + 3x + 5$. We apply the rule above. Each term gets their power dicreased by one, and the power multiplies the coefficient. So, the term $2x^3$ becomes $6x^2$ and so on. We have:

$$\begin{aligned} f'(x) &= 3 \cdot 2x^{3-1} - 2x^{2-1} + 3 \\ &= 6x^2 - 2x + 3. \end{aligned}$$

Example 4.2.2. Let $f(x) = x^{47} - 7x^{31} + 5x$. Then, the derivative of $f(x)$ is given by

$$f(x) = 47x^{46} - (7 \cdot 31)x^{30} + 5 = 47x^{46} - 217x^{30} + 5.$$

4.3 Product and quotient rules

The product and quotient rules are formulas that allow to compute the derivatives of products and quotients of functions that are differentiable.

Theorem 4.3.1. (Product Rule or Leibniz Rule) *Let $f(x)$ and $g(x)$ be differentiable. Then, the product function $(fg)(x) = f(x)g(x)$ is differentiable as well. The following formula holds for all x where f and g are differentiable*

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Proof. Let us consider a generic point x in the domain of f' and g' (i.e. where they are differentiable). We want to compute the derivative of the product. When x changes by h, $f(x)$ varies by $\Delta f = f(x+h) - f(x)$. Similarly for g we have $\Delta g = g(x+h) - g(x)$. So, to compute the limit

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h},$$

we can use the Δf and Δg above to get

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{(f(x)+\Delta f)(g(x)+\Delta g)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x)\Delta g+\Delta f g(x)+\Delta f\Delta g}{h}\\
&= \lim_{h\to 0}\frac{f(x)\Delta g}{h}+\lim_{h\to 0}\frac{\Delta f g(x)}{h}+\lim_{h\to 0}\frac{\Delta f\Delta g}{h}.
\end{aligned}$$

The first limit gives us $f(x)g'(x)$, while the second limit gives us $f'(x)g(x)$. We now just need to show that the third limit is zero. Toward this, observe that the limit can be split in a product of limits

$$\lim_{h\to 0}\frac{\Delta f\Delta g}{h} = (\lim_{h\to 0}\Delta f)(\lim_{h\to 0}\frac{\Delta g}{h}).$$

In the latter, $\lim_{h\to 0}\frac{\Delta g}{h} = g'(x)$ by definition, while $\lim_{h\to 0}\Delta f = 0$ by continuity of f (being differentiable f is also continuous). Therefore, the whole limit $\lim_{h\to 0}\frac{\Delta f\Delta g}{h}$ is zero, and we are done. □

A similar procedure shows the Quotient rule.

Theorem 4.3.2. (Quotient Rule)

Let f and g be differentiable and assume that g is nonzero. Then the quotient is differentiable as well. The following formula holds wherever f and g are differentiable.

$$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}.$$

Let us now consider some applications of these facts.

Example 4.3.3. Suppose we want to compute the derivative of the function $h(x) = (x^2-3)(2x+1)$. Then, since we know how to differentiate x^2-3 and $2x+1$, we can use the product rule to obtain the derivative of h. Here we set $f(x) = x^2-3$ and $g(x) = 2x+1$ in the product rule. We have

$$\begin{aligned}
h'(x) &= (\frac{d}{dx}(x^2-3))(2x+1)+(x^2-3)(\frac{d}{dx}(2x+1))\\
&= 2x(2x+1)+(x^2-3)2,
\end{aligned}$$

where in the last step we have used the rule to differentiate polynomials.

Example 4.3.4. Consider the function $f(x) = \frac{x^3-3x+1}{x^2+1}$. We can compute the derivative of this function by using the quotient rule. Observe that the domain of the function is $(-\infty,\infty)$ (why?).

Now, we apply the quotient rule to get

$$\begin{aligned}
\frac{d}{dx}(f(x)) &= \frac{d}{dx}(\frac{x^3-3x+1}{x^2+1}) \\
&= \frac{(\frac{d}{dx}(x^3-3x+1))\cdot(x^2+1)-(x^3-3x+1)\cdot(\frac{d}{dx}(x^2+1))}{(x^2+1)^2} \\
&= \frac{(3x^2-3)(x^2+1)-(x^3-3x+1)2x}{(x^2+1)^2} \\
&= \frac{3x^4-3x^2+3x^2-3-2x^4+6x^2-2x}{(x^2+1)^2} \\
&= \frac{x^4+6x^2-2x-3}{(x^2+1)^2}.
\end{aligned}$$

4.4 Derivatives of trigonometric functions

We compute now the derivative of the function $f(x) = \sin(x)$. From the definition of derivative, we need to evaluate the limit

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}.$$

Before evaluating the limit, recall the summation formula for sin: $\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x)$. Now, let us use this to compute the limit above.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} \\
&= \lim_{h\to 0}[\frac{\sin(x)\cos(h)-\sin(x)}{h}+\frac{\cos(x)\sin(h)}{h}] \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)}{h}+\lim_{h\to 0}\frac{\cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0}\sin(x)\frac{\cos(h)-1}{h}+\lim_{h\to 0}\cos(x)\frac{\sin(h)}{h} \\
&= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h},
\end{aligned}$$

where at the last step, we have moved $\sin(x)$ and $\cos(x)$ out of the limit because they do not depend on h (the limit is with respect to h!), and they are therefore constant as we let h go to zero. We are therefore left with the issue of evaluating the limits $\lim_{h\to 0}\frac{\cos(h)-1}{h}$ and $\lim_{h\to 0}\frac{\sin(h)}{h}$. We have the following result

Proposition 4.4.1. *The following equalities hold*

- $\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} = 0,$
- $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1.$

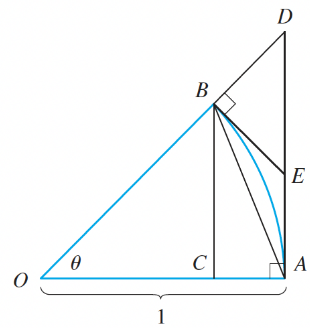


Figure 4.1: Figure showing the inequalities $\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$

Proof. We prove only the second limit, as the first can be obtained by a straightforward computation which involves multiplying by the $\frac{\cos(h)+1}{\cos(h)+1}$, and then utilizes the result from the second limit.

Suppose that $0 < \theta < \pi/2$. In the other cases one can proceed analogously by manipulating Figure 4.1. In fact, from the figure we see that $BC = \sin(\theta) < AB = \theta$, recalling the definition of arc and angle, and sin from trigonometry. From the previous inequality it follows that

$$\frac{\sin(\theta)}{\theta} < 1.$$

Moreover, $AD = \tan(\theta)$ again by the definitions in trigonometry. Therefore, it also holds that $AB = \theta < \tan(\theta) = AD$. But, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, and therefore $\theta < \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$. From $\theta < \frac{\sin(\theta)}{\cos(\theta)}$ we obtain $\cos(\theta) < \frac{\sin(\theta)}{\theta}$. So, we have the two inequalities

$$\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1.$$

By the Squeeze Theorem we can take the limits of $\cos(\theta)$ and 1 to obtain the limit of $\frac{\sin(\theta)}{\theta}$. Since

$$\lim_{\theta \to 0} \cos(\theta) = \lim_{\theta \to 0} 1 = 1,$$

it follows that $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$. □

4.5 Chain rule

The chain rule is a differentiation rule that applies to the cases where we want to obtain the derivative of a function that is the composite of two smaller functions. Here the main thing is to individuate a way of decomposing a function into pieces that we know how to differentiate.

Example 4.5.1. Consider the function $h(x) = \sqrt{x^2 + 1}$. Suppose that I wanted to compute the output of h evaluated at the point $x = 0$, using a calculator. In this case, I should first take $x = 0$ and square it, and sum 1. In other words, the first thing to do is $0^2 + 1$. Once I get my result, then I can simply take it and use the square root to get $\sqrt{0^2 + 1}$.

The fact that our way of computing the output of h consists of applying two procedures, first x^2+1 and then using $\sqrt{}$, indicates that f is a composite function. Where we first have $g(x) = x^2+1$, and then $f(x) = \sqrt{x}$. In fact, by definition of composite $f(g(x))$, we have that

$$f(g(x)) = f(x^2+1) = \sqrt{x^2+1}.$$

The function g is called *inner* function, and the function f is called *outer* function. The chain rule gives us a way of computing the derivative of h, from the derivatives of g and f.

Theorem 4.5.2. (Chain Rule) *Let f and g be differentiable functions, and let $h(x) = f(g(x))$ be a composite function. Then, the derivative of h is obtained through the rule*

$$\frac{d}{dx}(h(x)) = f'(g(x))g'(x),$$

where $f'(g(x))$ is the derivative of the outer function, evaluated at $g(x)$.

Example 4.5.3. We now go back to the example of $h(x) = \sqrt{x^2+1}$, and we compute the derivative. We already know how to decompose h into an inner and an outer function. We have found that $f(x) = \sqrt{x}$ is the outer function, and $g(x) = x^2+1$ is the inner function. Now, let us take the derivative of f and g separately. We have

$$f'(x) = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}},$$

and also

$$g'(x) = \frac{d}{dx}(x^2+1) = 2x.$$

Now we can apply the chain rule by evaluating $f'(x) = \frac{1}{2\sqrt{x}}$ at $g(x) = x^2+1$ (i.e. replacing x with $g(x)$!) and then multiplying it all by $g'(x) = 2x$. We have

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= \frac{1}{2\sqrt{x^2+1}}2x \\ &= \frac{x}{\sqrt{x^2+1}}. \end{aligned}$$

We now prove the chain rule.

Proof. (Chain rule)

Recall that one of the way to define the derivative is that $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$, where Δy represents the variation of the function corresponding to the variation of the argument Δx. We used this definition to obtain the instantaneous velocity of a moving object. If the derivative exists, then we have that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} - f'(x) = 0. \tag{4.1}$$

So, we can define a small quantity ϵ such that $\epsilon = \frac{\Delta y}{\Delta x} - f'(x)$, and $\epsilon \to 0$ as $\Delta x \to 0$. So,

$$\Delta y = f'(x)\Delta x + \epsilon \Delta x. \tag{4.2}$$

This equation holds for any x where f is differentiable.

Consider now a composite function $y(x) = f(g(x))$. We call $u = g(x)$ the input of f. So, we have $y = f(u)$. We consider the derivative of the composite at $x = a$, and set $b = g(a)$. If we have a variation in x, which we denote by Δx, then $g(x)$ varies by a quantity that we denote by Δu (since we put $u = g(x)$). But if u varies, then the argument of f is varying, which means that f varies as well. We call this variation Δy. Then, using Equation (4.2) applied to $g(x)$ and $f(u)$, we can write

$$\Delta u = g'(a)\Delta x + \epsilon_1 \Delta x, \tag{4.3}$$

and

$$\Delta y = f'(b)\Delta x + \epsilon_2 \Delta x. \tag{4.4}$$

Substituting Equation (4.3) into Equation (4.4), we obtain

$$\Delta y = (f'(b) + \epsilon_2)(g'(a) + \epsilon_1)\Delta x.$$

Therefore, dividing boths sides by Δx we find

$$\frac{\Delta y}{\Delta x} = (f'(b) + \epsilon_2)(g'(a) + \epsilon_1).$$

Taking the limit as Δx goes to zero, both ϵ_1 and ϵ_2 go to zero, as in Equation (4.2). Therefore we find

$$\frac{dy}{dx} = f'(b)g'(a).$$

Since $b = g(a)$ by assumption above, we have completed the proof of the chain rule. □

4.6 Derivatives of Exponentials and Logarithms

We give now some rules of differentiation without proof.

- $\frac{d}{dx}(e^x) = e^x$.
- $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$.
- $\frac{d}{dx}(b^x) = b^x \ln(b)$.
- $\frac{d}{dx}(\log_b(x)) = \frac{1}{x \ln(b)}$.

Example 4.6.1. We want to compute the derivative of the function $h(x) = e^{\sin(x)}$. The function h is a composite function obtained by composing the sine function and the exponential function. We can therefore write $f(x) = e^x$ and $g(x) = \sin(x)$, with $h(x) = f(g(x))$. The derivative of $f(x)$ is e^x, as seen above. The derivative of $g(x)$ is $\cos(x)$ as we have showed before. So, we have to apply the chain rule. We have

$$\frac{d}{dx}(e^{\sin(x)}) = e^{\sin(x)} \frac{d}{dx}(\sin(x)) = e^{\sin(x)} \cos(x).$$

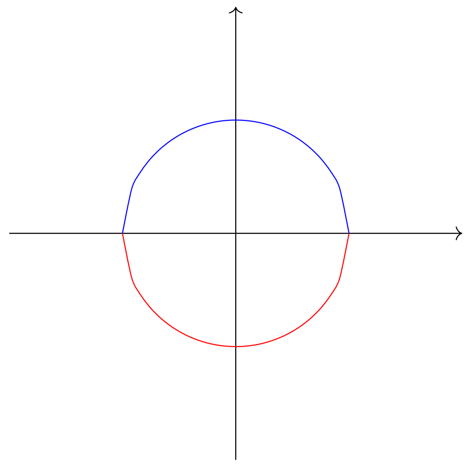


Figure 4.2: Plot of the curve $x^2 + y^2 = 3$.

4.7 Implicit Differentiation

Implicit functions arise when we have relations between x and y that are not simply written in the form of a function as $y = f(x)$, which is what we have been considering so far. These equations arise when we have relations of type

$$x^2 + y^2 = 3 \quad (4.5)$$

$$x^3 + y^3 = 6xy. \quad (4.6)$$

These are curves in the plane, and explicitly writing y as a function of x is not simple. Equation (4.5) is shown in Figure 4.2. This is obviously not a function (it fails the vertical line test!). However, we can isolate two components (the red and the blue ones) where the curve does indeed define a function. So, while globally we do not have a function, we locally (on some parts of the curve have a well defined function).

Our scope is to understand how to differentiate functions that arise in this way. We assume, here, that around a chosen point of the curve we are able to explicitly write y as a function of x. This is in general very difficult to do. For instance, while for Equation (4.5) we can find two curves, $y_1 = \sqrt{3 - x^2}$ (the blue on top) and $y_2 = -\sqrt{3 - x^2}$ (the red below), for the curve in Equation (4.6) (the folium of Descartes), this is quite complicated.

However, we can proceed using a trick as discussed below. Here we assume that around a point of interest, we can indeed write our y variable as a function of x.

Example 4.7.1. We want to compute the derivative of y as a function of x at the point $(1, \sqrt{2})$. First, differentiate the whole equation with respect to x, assuming that around our point y is expressible as a function of x. We have

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(3),$$

where the right hand side is zero (the derivative of a constant!), and the left hand side is the derivative of the function $f(x) = x^2$, which we know gives $2x$, and the derivative of the function

$y(x)^2$. Now, $y(x)^2$ is a composite function obtained by composing $y(x)$ and using the power (by 2) function. Therefore, $\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$ by the chain rule. We have found:

$$2x + 2y\frac{dy}{dx} = 0.$$

Now, we can explicitly write the derivative $\frac{dy}{dx}$ as

$$\frac{dy}{dx} = -\frac{2x}{2y} = -x/y.$$

Substituting the values $x = 1$ and $y = \sqrt{2}$ of our point, we find the derivative of y evaluated at the point $(1, \sqrt{2})$.

Example 4.7.2. We want to compute the derivative of y for the folium of Descartes $x^3+y^3 = 6xy$. Also, we want to use this result to get the tangent to the folium of Descartes at the point $P \equiv (3, 3)$.

We differentiate both sides of the equation $x^3 + y^3 = 6xy$ with respect to x, considering y as a function of x. We get

$$3x^2 + 3y^2 \cdot y' = 6y + 6xy',$$

where we have used the chain rule for y^3, and the Leibniz rule (product rule) for $6xy$. Therefore, dividing the previous equation by 3, we have

$$x^2 + y^2 \cdot y' = 2y + 2xy',$$

and we can explicitly write for y':

$$(y^2 - 2x)y' = 2y - x^2,$$

which gives

$$y' = \frac{2y - x^2}{y^2 - 2x}.$$

To know the tangent line at P, we can substitute the coordinates of P into the equation of y' to obtain the slope of the tangent. We get

$$y' = -1.$$

To write the equation of the line, recall that the tangent has an equation of type $y = mx+q$, where m is the slope, which we have just found to be $m = y' = -1$. To obtain q, substitute the values of P inside $y = -x + q$ and get $q = 6$. So, the equation of the tangent line is $y = -x + 6$.

Logarithmic Differentiation

We want to differentiate the function $y = \frac{x^3\sqrt{x^2+1}}{(3x^2+2)^5}$.

We take the logarithm of both sides of the equation:

$$\ln(y) = \ln(\frac{x^3\sqrt{x^2+1}}{(3x^2+2)^5}).$$

Using the properties of logarithms, the RHS becomes quite simple, and we get

$$\ln(y) = 3\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(3x^2+2).$$

We now use implicit differentiation (and the rule of differentiation of logarithms) to get

$$\frac{y'}{y} = \frac{3}{x} + \frac{1}{2}\frac{2x}{x^2+1} - 5\frac{6x}{3x^2+2}.$$

This means that

$$y' = y[\frac{3}{x} + \frac{x}{x^2+1} - 5\frac{6x}{3x^2+2}].$$

Substituting the value of $y = \frac{x^3\sqrt{x^2+1}}{(3x^2+2)^5}$ back in the previous equation, we get the derivative of y:

$$y' = \frac{x^3\sqrt{x^2+1}}{(3x^2+2)^5}[\frac{3}{x} + \frac{x}{x^2+1} - 5\frac{6x}{3x^2+2}].$$

4.8 Applications of Derivatives

Example 4.8.1. We consider now an application of differentiation to chemical reactions. Namely, we consider how to determine the variations of substances in a chemical reaction. Chemical reactions generally have reactants that very over time and the dynamics of such variation is of great interest in chemistry. We will see that such a situation can be studied with the tools we have learned so far.

The reaction obtained by burning propane is given by

$$C_3H_8 + 5 \cdot O_2 \longrightarrow 3 \cdot CO_2 + 4 \cdot H_2O + \text{Heat}.$$

All the quantities in the chemical equation vary with respect to time, during the reaction. In such a situation, chemists are interested in finding the rate of reaction, wich is the instantaneous variation of some of the reactants in the chemical equation.

For instance, suppose we want to know the rate of reaction of the propane C_3H_8 in the chemical equation above. Then, we need to compute the derivative

$$\frac{d}{dt}(C_3H_8) = ?$$

Since the amount of propane decreases during the reaction, the derivative will be negative. Similarly, the oxygen will have negative derviative. On the other hand, quantitites on the RHS of the chemical reaction will increase, and their derivative will be positive.

Each time 3 moelucles of CO_2 and 4 molecules of H_2O are produced, one molecule of propane and 5 molecules of O_2 are consumed. This means that the rates will be related by the equations

$$-\frac{d}{dt}(C_3H_8) = -\frac{1}{5}\frac{d}{dt}O_2 = \frac{1}{3}\frac{d}{dt}CO_2 = \frac{1}{4}\frac{d}{dt}H_2O.$$

Therefore, if we can measure the variation of one of the quantities, we can obtain the derivatives of the reactants.

Example 4.8.2. Radioactive substances spontaneously decay by emitting radiation. This process happens with a certain probability which depends on the substance considered. The probability of a mass of substance m emitting radiation depends in a directly proportional manner also on the amount of substance, i.e. how large m is. This is because the larger m, the higher the number of radioactive atoms are present, and therefore the higher the probability of radioactive emission.

Experimentally, it has been found that the probability of decay is constant with respect to time. Let us call this constant k. If $m(t)$ indicates the mass of the radioactive substance at time t, since the probability of emitting radiation at time tt is proportional to $m(t)$ (by k), it follows that the variation of $m(t)$ is directly proportional to m as well, because if atoms that emit radiation get transformed, and will not be counted in $m(t)$ anymore.

Therefore, we have that

$$\frac{dm(t)}{dt} = -km(t),$$

where the negative sign is due to the fact that the radioactive substance discreases.

We have to find a function $m(t)$ such that its derivative is proporitonal to $m(t)$ by a factor of $-k$. We know that the exponential function has this property! In fact, we can simply see that setting

$$m(t) = Ae^{-kt},$$

we find that the required property is satisfied, where A is just a constant. Taking time $t = 0$ we see that $m(0) = Ae^0$, which shows that $A = m(0)$ is just the initial mass of the substance.

To summarize, given a certain amount of radioactive substance m_0 at time $t = 0$, we can compute how much radioactive substance is left at a subsequent time, say $t = 1hr$ by using the function $m(t) = m_0e^{-kt}$, upon knowing the radioactive decay constant k, which is determined experimentally.

Example 4.8.3. We consider now an example from biology.

Denote by $n = f(t)$ the number of individuals in a population (e.g. number of predators in some environmnet). Suppose that the population $f(t)$ changes according to some law $f(t) = n_0 2^t$, where n_0 is the initial population at time $t = 0$. Such a law roughly relates the growth of bacteria in some nutrient medium.

Now, let us say that we want to understand the variation of the population at time $t = 10$, i.e. after 10 hours. In order to do that, we need to compute the instantaneous variation of $n = f(t)$ with respect to time, and evaluate it at time $t = 10$. We know that

$$f'(t) = \frac{d}{dt}(n_0 2^t) = n_0 2^t \cdot \ln(2).$$

Evaluating at time $t = 10$, we obtain the variation after 10 hours, which is given by $n_0 2^{10} \ln(2)$. So, starting with around $n_0 = 10$ bacteria, after 10 hours we will have a growth rate of around ≈ 7000.

Example 4.8.4. Assume that the trajectory of a particle is given by the equation:

$$s = f(t) = t^3 - 6t^2 + 9t,$$

where t indicates time, and s indicates the position (along one dimension, e.g. a distance), where time is given in seconds and space is given in meters. Determine the following:

1. Velocity of the particle as a function of time.

2. Acceleration of the particle as a function of time.

3. Velocity at time $t = 2$ sec and $t = 4$ sec.

4. Acceleration at time $t = 2$ sec and $t = 4$ sec.

5. When the particle is moving forward, and when the particle is at rest.

6. The total distance traveled by the particle in the first 5 seconds.

1. To find the velocity, we need to compute the derivative of s. This means that

$$v(t) = s'(t) = 3t^2 - 12t + 9.$$

2. To find acceleration, we need to take the second derivative. In other words, we have to differentiate again the first derivative. We get

$$a(t) = s''(t) = 6t - 12.$$

3. To obtain the velocity at time $t = 2$, we plug $t = 2$ in the expression for the velocity. We get

$$v(2) = -3\frac{m}{s}.$$

Similarly, at 4 seconds we get

$$v(4) = 9\frac{m}{s}.$$

4. To obtain the acceleration at $t = 2$ and $t = 4$, as before, we need to plug in the acceleration function. We have

$$a(2) = 0\frac{m}{s}, \quad a(4) = 12\frac{m}{s}.$$

5. To detemine when the particle is moving forward, we have to find when the particle has positive velocity. To achieve this, we have to determine the sign of $v(t) = 3t^2 - 12t + 9$. Observe that $v(t) = 3(t-1)(t-3)$. Therefore, its sign is determine by when the product $(t-1)(t-3)$ is positive, and when it is negative. A product is positive only when both signs of the terms are agreeing (either both positive, or both negative). When $x \leq 1$, both signs are negative, and therefore $v(t) \geq 0$. When $1 < t < 3$, the signs do not agree, since $(t-1)$ is positive and $(t-3)$ is negative, so $v(t) < 0$. When $t \geq 3$, finally, the terms are both positive, and therefore $v(t) \geq 0$. So, to summarize, the particle moves forward between 0 to 1 seconds, and then again from 3 seconds on. Morevoer, we know that $v(t) = 0$ for $t = 1$ and $t = 3$, so that at those times it is at rest.

6. To determine the total distance traveled, we need to consider that (from part 5.) the particle goes forward until $t = 1$, it goes back again between $t = 1$ and $t = 3$, and it moves forward again. Therefore, we have to sum the distance traveled during each portion of the trajectory separately. We $|f(1) - f(0)| = 4m$. Then $|f(1) - f(3)| = 4m$, and lastly $|f(5) - f(3)| = 20m$.

Example 4.8.5. Consider the flow of blood in a blood vessel. We assume that the blood vessel is approximated as a long cylinder with length l and radius R. In general, l is much bigger than R. The velocity of the blood flow is depends on how close to the central axis of the vessel we are, due to friction between the blood and the vessel. The velocity is described by the *Law of laminar flow*, given by

$$v = \frac{P}{4\eta l}(R^2 - r^2),$$

where r is the distance between a point and the central axis, P is the pressure and η is the viscosity of blood. Therefore, the velocity is a function of r, assuming that P and η are simply constants. The gradient of velocity as a function of the distance from the central axis is given by

$$\frac{dv}{dr} = -\frac{P}{2\eta l} r,$$

and it is simply the derivative of the function v with respect to r.

In a human artery we can take $\eta = 0.0027$, $R = 0.008$ cm, $l = 2$ cm, and $P = 4000$ dynes/cm^2. So, we get at $r = 0.002$ cm a value for the velocity of $v(0.002) = 1.11$ cm/sec, and the velocity gradient at that point is of about $v'(0.002) = -74$ (cm/sec)/cm.

Example 4.8.6. Newtpon's law of cooling states that the temperature variation (in time) of an object at temperature T in an environment with temperature T_s, is given by

$$\frac{dT}{dt} = k(T - T_s),$$

where k is a constant that is typical of the material of the object.

Suppose we know that a bottle of iced tea at room temperature (21 °C) is placed in a refrigerator with temperature of 5 °C, and reaches the temperature of 15 °C after 30 minutes. What will the temperature be after 30 more minutes?

Chapter 5

Main results on derivatives with applications

5.1 Maxima and minima

In practice, we are interested in several cases to understand the maximum and minimum value that certain quantities can attain. We will see that caclulus gives us several tools that can be used to obtain this information.

First, we define what maximum and minimum values mean.

Definition 5.1.1. Let c be a number in the domain D of f such that either of the following statements hold:

- $f(c) \geq f(x)$ for all x in D.
- $f(c) \leq f(x)$ for all x in D.

Then, in the first case we say that $f(c)$ is a global maximum for f, and in the second case we say that $f(c)$ is a *global minimum* for f. We sometimes also say that they are absolute maxima and minima, or simply *extreme values*.

We also have the following definition.

Definition 5.1.2. Let c be a point of the domain D of f. Assume that there exist an interval $[a, b]$ containing c such that either of the following holds:

- $f(c) \geq f(x)$ for all x in $[a, b]$.
- $f(c) \leq f(x)$ for all x in $[a, b]$.

Them, in the first case we say that $f(c)$ is a *local maximum*, and in the second case we say that $f(c)$ is a *local minimum*. The meaning of this definition is that $f(x)$ is smaller (or larger) than $f(c)$ when we take x close to c.

Remark 5.1.3. A global maximum or minimum value is always also a local maximum or minimum value. The converse is not true. This is quite general: Global properties are also true locally, but

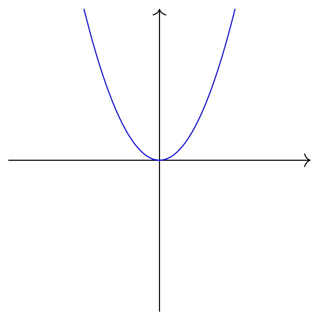


Figure 5.1: Plot of function $f(x) = x^2$. The minimum is found at $x = 0$.

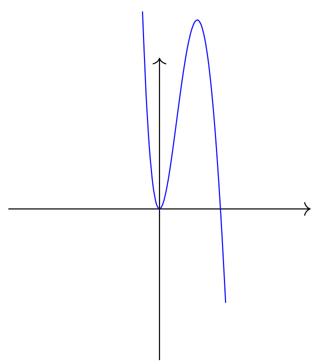


Figure 5.2: Plot of a polynomial function that has local maximum and local minimum, but not global.

local properties are not true globally. For instance, if I own the tallest building in town, it does not mean that I own the tallest building on earth. However, if I happen to own the tallest building in the world, for sure I own also the tallest building in town.

Example 5.1.4. The function $f(x) = x^2$ has a global minimum at $x = 0$, as it is clear from the graph of it. However, this function does not have maxima (why?).

Example 5.1.5. The function $f(x) = \sin(x)$ has infinitely many global maxima and minima. They are those angles such that $\sin(x) = 1$ and $\sin(x) = -1$, respectively. They repeat periodically, as it is found in precalculus courses.

Example 5.1.6. Take the polynomial function $f(x) = 3x^4 - 16x^3 + 18x^2$. Then, as shown in the graph of it, this function has a local minimum, a local maximum, but not global maxima and minima.

We have now the following very imporant result, which we state without proof.

Extreme value theorem.

Theorem 5.1.7. *If f is a continuous function defined over the closed interval $[a, b]$, then f has a global maximum and a global minimum.*

We will refer to this theorem as the *Extreme Value Theorem*, or EVT for short.

Now, the question that arises is whether there are more efficient methods to find extreme values, rather than graphing the function and guessing. We develop now the tools necessary to achieve this.

Theorem 5.1.8. *If f has a local maximum or minimum at c, and $f'(c)$ exists, then we have*

$$f'(x) = 0.$$

Proof. We will prove the statement for the case when $f(c)$ is a local maximum. The proof for the other case is substantially the same, and it is left to the reader as an exercise. From the definition of local maximum, we know that around c, $f(c) \geq f(x)$. Around c here means that in some interval $[a, b]$ containing c (and contained in the domain of f), we have this property. Therefore, in particular, when considering elements x close enough to c we have the inquality

$$\frac{f(c+h) - f(c)}{h} \geq 0$$

whenerver $h \geq 0$ and $c + h$ is in $[a, b]$ (i.e. h is small enough). Similarly, for $h < 0$ such that $c + h$ is in $[a, b]$, we have also Therefore, in particular, when considering elements x close enough to c we have the inquality

$$\frac{f(c+h) - f(c)}{h} \leq 0.$$

Upont letting h go to zero from the right and the left (respectively), the previous inequalities give us that

$$\lim_{x \to 0^+} \frac{f(c+h) - f(c)}{h} \geq 0,$$

and

$$\lim_{x \to 0^-} \frac{f(c+h) - f(c)}{h} \leq 0.$$

Since f is differentiable at c, the right and left limits must coincide, and therefore

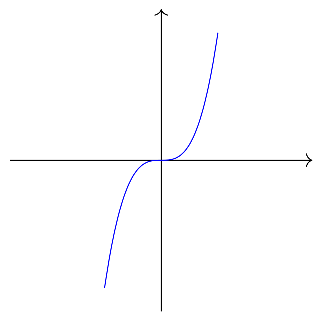
$$\lim_{x \to 0} \frac{f(c+h) - f(c)}{h} = 0.$$

This means that $f'(c) = 0$, which is what we wanted to prove. □

The previous theorem, which is due to Fermat, has a very natural geometric interpretation. In fact, $f'(c) = 0$ means that the tangent line is horizontal. But this is intuitively clear, since where f attains a maximum or minimum the tangent line cannot be skew, provided that it exists!

The following example shows that if $f'(c) = 0$, it is not generally true that the point c corresponds to a maximum or minimum.

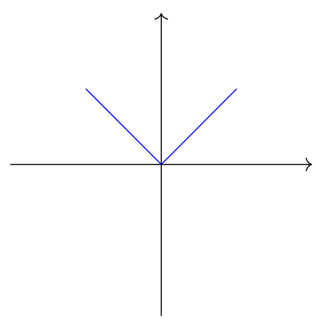
Example 5.1.9. Let $f(x) = x^3$. The graph of this function looks like



We know that $f'(x) = 3x^2$ has a zero at $x = 0$. So that $f'(0) = 0$. However, it is clear that $f(0)$ is not an extreme value.

The following example shows that a continuous function f can attain its extreme value without the derivative being zero, when it is not differentiable at a point.

Example 5.1.10. Consider the function $f(x) = |x|$. Then, the graph looks like:



It is clear that $x = 0$ is a minimum for $f(x)$. However, we know that this function is not differentiable at zero, and therefore $f'(0)$ does not exist, which in particular means that it is not equal to zero.

Definition 5.1.11. A *critical number* (or critical point) for a funtion f is a number c in the domain of f where one of the two possibilities occurs:

- $f'(c)$ exists, and $f'(c) = 0$.
- $f'(c)$ does not exist.

These results give us a procedure to obtain maxima and minima for continuous functions on a closed interval.

Method 5.1.12 (Closed Interval Method). *Let $f(x)$ be a continuous function defined over the interval $[a, b]$. Then, to obtain the maxima and minima of f, we perform the following procedures.*

- *Find the critical numbers of f in (a, b).*
- *Find the values of f at the endpoints a and b.*
- *Compare all the values obtained in the previous two steps. The largest of them all will be the global maximum, and the smallest of them all will be the global minimum.*

Example 5.1.13. Let $f(x) = x^3 - 2x^2 + 7$ be defined on $[-2, 2]$. We want to find the maxium and minimum of this function. Since it is continuous, we can use the Closed Interval Method. We first need to find the critical numbers of f in the interval $[-2, 2]$. Compute $f'(x)$:

$$f'(x) = 3x^2 - 4x.$$

Now, we need to solve the equation $f'(x) = 0$, which means $3x^2 - 4x = 0$. We have

$$f'(x) = x(3x - 4) = 0$$

gives solutions $x = 0$ and $x = 4/3$. So, now we compute the function f on the points -2, 2, 0 and 4/3. We get

- $f(-2) = -8 - 8 + 7 = -9$.
- $f(0) = 7$.
- $f(4/3) = 64/27 - 2 \cdot 16/9 + 7 \approx 5.814$.
- $f(2) = 8 - 8 + 7 = 7$.

Therefore, we have found that $f(-2)$ is the global minimum, and that $f(0)$ and $f(2)$ are both global maxima.

5.2 The Mean Value Theorem

We now obtain several important applications of differentiation, including the celebrated Mean Value Theorem (MVT). Before proving the MVT, we introduce and show Rolle's Teheorem.

Theorem 5.2.1 (Rolle). *Let f be a function that satisfies:*

1. *f is continuous on $[a, b]$.*
2. *f is differentiable on (a, b).*
3. *$f(a) = f(b)$.*

Then, there exists a number c in (a, b) such that $f'(c) = 0$.

Proof. We have to distinguish some cases. First, consider the case where $f(x) = k$ is a constant. Then, of course the derivative of $f(x)$ is zero in the whole interval and a c as in the statement surely exists.

Suppose now that there exists at least a value x in (a, b) such that $f(x) \neq f(a)$ (or we would be in the previous case). Let us consider now two subcases of this situation. Suppose first that $f(x) > 0$ for some number in (a, b). Applying the Extreme Value Theorem (i.e. Theorem 5.1.7) we find that f attains its maximum in $[a, b]$. Since $f(x) > f(a) = f(b)$ at some point, the maximum cannot be reached at a or b. Therefore, this point which we call c lies in (a, b). By Fermat's Theorem (Theorem 5.1) $f'(c) = 0$. Similarly, if $f(x) < f(a)$ in some point in (a, b), then by considering the minimum, Theorem 5.1.7 and Theorem 5.1.7, we find that $f'(c) = 0$ again. This completes the proof. □

Example 5.2.2. Show that the equation $x^3 + x - 1 = 0$ has exactly one solution.

First, we want to show that at least one solution exists. Set $f(x) = x^3 + x - 1$. We apply the Intermediate Value Theorem to 0 and 1. In fact, $f(0) = -1 < 0$ and $f(1) = 1 > 0$. So, there exists at least one solution. Now we need to show that it is unique.

Suppose that there exist two solutions, which we call $x = a$ and $x = b$. Then, we have $f(a) = f(b) = 0$. Since all the hypotheses of Rolle's Thoerem are satisfied in $[a, b]$, we have that there exists a point c in (a, b) such that $f'(c) = 0$. But $f'(x) = 3x^2 + 1 > 0$, so it is never going to be zero. We have found a contradiction, which means that our assumption that two solutions could exist was wrong. There cannot be two distinct solutions.

We now state and prove the Mean Value Theorem, also known as Lagrange Theorem in honor of Lagrange.

Theorem 5.2.3 (MVT). *Let f be a function that satisfies the following conditions.*

1. *f is continuous on the interval $[a, b]$.*

2. *f is differentiable in (a, b).*

Then, there exists a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof. Define the new function $h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$. It is easy to see that $h(x)$ satisfies all the assumptions of Rolle's Theorem. In fact, $h(x)$ consits of a sum of $f(x)$ which is continuous, and a polynomial function, which is continuous as well. Therefore, hypothesis 1. in Rolle's Theorem holds. Morevover, f is differentible and any polynomial is differentiable, so that 2. holds as well. Lastly, $h(a) = 0 = h(b)$, and the last assumption of Rolle's Theorem also holds. By applying Rolle's Theorem to $h(x)$, we get that there exists a point c in (a, b) such that $h'(c) = 0$. Since $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$, it follows that $h'(c) = 0$ implies $f'(c) = \frac{f(b)-f(a)}{b-a}$, which completes the proof. □

As an application we will see that the only functions that have vanishing derivatives over a full interval are those functions that are constant over the same interval.

Theorem 5.2.4. *If $f'(x) = 0$ over the interval (a, b), then f is a constant function on (a, b).*

Proof. Let x_1 and x_2 be two numbers taken from the interval (a, b). Suppose that $x_1 < x_2$. Since f is differentiable on (a, b), then it will be continuous and differentiable over $[x_1, x_2]$ and we can apply the Mean Value Theorem. We therefore find c in (x_1, x_2) such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Since $f'(c) = 0$, it follows that $f(x_1) = f(x_2)$. Therfore, f takes the same value on any pair of two numbers chosen from (a, b), which means that it is a constant function over (a, b). □

Theorem 5.2.5. *If $f'(x) = g'(x)$ over the interval (a, b), then $f(x) = g(x) + k$ for some constant $k \in \mathbb{R}$.*

Proof. Set $F(x) = f(x) - g(x)$. Then, $F'(x) = f'(x) - g'(x) = 0$. Applying Theorem 5.2.4 we find that $F(x)$ is constant, i.e. $F(x) = k$ fir some number k. It follows that $f(x) - g(x) = k$ from which we complete the proof. □

	$(-\infty,-1)$	$(-1,0)$	$(0,2)$	$(2,\infty)$
x	$-$	$-$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$	$+$

Table 5.1: Signs to determine where $f(x)$ is increasing or decreasing.

5.3 Getting information on f through its derivative

We now find more applications of the Mean Value Theorem. In this section we will see how the derivative of f, and higher derivatives well, can be used to obtain information regarding the behavior of the function f.

First, we will see that the derivative of a function tells use where the function increases or decreases (recall the definition of increasing or decreasing function!).

Theorem 5.3.1 (Increasing/Decreasing Test). *If $f'(x) > 0$ on an interval, then f is increasing on that interval. If $f'(x) < 0$, then it is decreasing.*

Proof. Let x_1 and x_2 be two numbers in the interval with $x_1 < x_2$. Using the MVT we find a point c between x_1 and x_2 such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

which also can be written as

$$f'(c)(x_2 - x_1) = f(x_2) - f(x_1).$$

Since $f'(x) > 0$ on the whole interval, we find that $f'(c) > 0$. So, $f'(c)(x_2 - x_1) > 0$, which means that $f(x_2) - f(x_1) > 0$. This gives $f(x_2) > f(x_1)$ when $x_2 > x_1$. We have therefore shown that f is increasing. To show that f is decreasing when $f'(x) < 0$ on the interval is substantially the same proof, and we leave it to the reader. □

Example 5.3.2. Determine where the function $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 + 5$ is increasing, or decreasing.

We need to compute the derivative of f, and then study the sign of it. We have

$$f'(x) = x^3 - x^2 - 2x = x(x-2)(x+1).$$

So, to understand the sign of $f'(x)$, we need to consider where the product of the three terms is larger than zero. We have that $x - 2 \geq 0$ when $x \geq 2$, and $x + 1 \geq 0$ when $x \geq -1$. The first term of course is just $x \geq 0$. We therefore get the table of signs as in Table 5.3.2

Computing the product of the signs, we get that $f'(x) < 0$ in $(-\infty, -1)$, it is positive in (-1.0), negative again in (0.2), and then positve in $(2, \infty)$. So, f is decreasing in $(-\infty, -1)$, increasing in (-1.0), decreasing in (0.2), and increasing in $(2, \infty)$.

Method 5.3.3 (First Derivative Test). *Suppose that c is a critical point of a differentiable (around c) function f. Then, the following facts hold.*

	$[0, 2\pi/3]$	$(2\pi/3, 4\pi/3)$	$(4\pi/3, 2\pi)$
$f'(x)$	$+$	$-$	$+$

Table 5.2: The signs of $f'(x)$.

1. *If f' changes from positive to negative at c, then f is a local maximum.*
2. *If f' changes from negative to positive at c, then f is a local minimum.*
3. *If f' does not change sign around c (either is all positive or all negative), then f has neither a local maximum nor a local minimum.*

Example 5.3.4. Find the local maxima and minima of $f(x) = x + 2\sin(x)$ on the interval $[0, 2\pi]$.

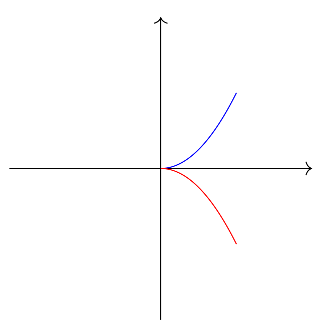
First of all, we need to find the critical points of f, since we know that if a point is a local maximum or a local minimum it will be a point where the derivative either does not exist, or where the derivative is zero. So, we have

$$f'(x) = 1 + 2\cos(x),$$

which means that the derivative always exists, and the critical points are those points where $f'(x) = 0$. This means that these points satisfy $\cos(x) = -1/2$. In $[0, 2\pi]$, the points x that solve the equation $f'(x) = 0$ are $x = 2\pi/3$ and $x = 4\pi/3$. We now need to determine the sign of f' around these two points to understand whether they are points of max, min or neither of them, using the First Derivative Test. Around $x = 2\pi/3$ we have that $\cos(x)$ is first larger than $-1/2$, meaning that $f'(x) > 0$, and then it becomes smaller than $-1/2$, meaning that $f'(x) < 0$. So, using the First Derivative Tests, we find that $x = 2\pi/3$ is a local maximum. For $x = 4\pi/3$ we have all the way around that $f'(x)$ is first negative and then positive, indicating that this is a local minimum. We can summarize this using a table of signs as in Table 5.3.4

Definition 5.3.5. We say that a function is *concave upward* in an interval, if the tangents to the graph of the function in that interval lie all below the graph. We say that it is *concave downward* if the tangent lines lie all above the graph.

The example below shows a function that is concave upward (in blue), and a function that is concave downward (in red). Draw the tangent lines to relate what you obtain to the definition.

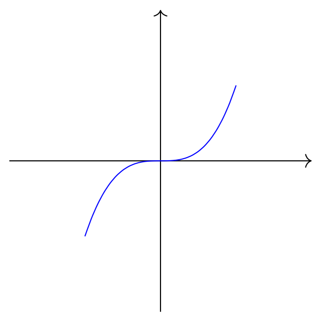


Method 5.3.6 (Concavity Test). 1. *If $f''(x) > 0$ on an interval, then the function f is concave upward on the interval.*

2. *If $f''(x) < 0$ on an interval, then the function f is concave downward on the interval.*

Definition 5.3.7. A continuous function f is said to have an inflection point at x if f changes its concavity at that point.

An example of inflection point (at $x = 0$) is seen in the graph below.



Method 5.3.8 (Second Derivative Test). If $f''(x)$ is continuous around a point c, then we can use the following criteria to find max and min.

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

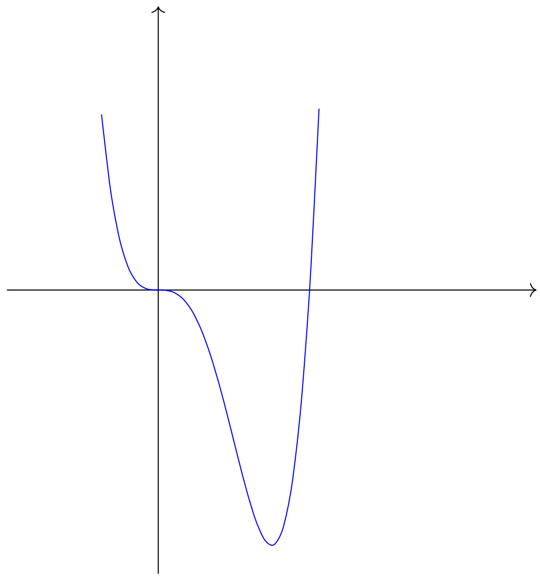
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example 5.3.9. Consider the curve $y = x^4 - 4x^3$. We want to analyze the maxima and minima of y, inflection points, concavity.

Since $y = f(x)$ is a polynomial, all its derivatives are continuous. So, to obtain maxima and minima we can use the Second Derivative Test. We have

$$\begin{aligned} f'(x) &= 4x^3 - 12x^2 \\ f''(x) &= 12x^2 - 24x = 12x(x-2). \end{aligned}$$

Since $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$, the critical points of f, i.e. where $f'(x) = 0$ are the points where $4x^2 = 0$, or $x - 3 = 0$. In other words, the critical points are $x = 0$ and $x = 3$. To determine whether these points are max or min, we use the Second Derivative Test and plug them in the form for $f''(x)$. This gives us $f''(0) = 0$, and $f''(3) = 36 > 0$. The test does not produce any result for $x = 0$, but it gives us that $x = 3$ is a point of local minimum. However, the sign of $f''(x)$ passes from negative to positive at the point $x = 0$, meaning that the Concavity Test says that f has concavity upward before 0 and downward after 0. This means that $x = 0$ is an inflection point. Moreover, the concavity changes again to positive after $x = 3$, as it can be seen by studying the sign of $f''(x)$. This allows us to sketch the function.



Example 5.3.10. Let us use what we know so far to sketch the graph of the function $f(x) = e^{\frac{1}{x}}$.

Since $f(x)$ has domain given by all $x \neq 0$, we need to sketch f everywhere but at $x = 0$. We have that

$$\lim_{x \to 0^+} f(x) = \infty,$$

since $1/x \to \infty$ as $x \to 0$ from the positive side. Since $1/x \to -\infty$ as $x \to 0$ from the negative side, we also have

$$\lim_{x \to 0^-} f(x) = 0.$$

So, $x = 0$ is a vertical line that is a vertical asymtptote on the right for $f(x)$. Let us now consider the horizontal asymptotes.

$$\lim_{x \to \infty} f(x) = 1,$$

since $1/x \to 0$ as $x \to \infty$. Also,

$$\lim_{x \to -\infty} f(x) = 1,$$

again because $1/x \to 0$ as $x \to -\infty$. Now, we know how $f(x)$ behaves at the "boundaries" of its domain. We need to fill the gap in between. To do so, we need to understand how f increases/decreases and the concavity of $f(x)$. We have

$$f'(x) = -\frac{e^{\frac{1}{x}}}{x^2}.$$

This means that $f'(x) < 0$ for all x in its domain, since both numerator and denominator are always positive. So, f is always decreasing.

Lastly, for the concavity, we need to comopute f''. We have

$$f''(x) = \frac{e^{\frac{1}{x}}(2x+1)}{x^4}.$$

The only term in $f''(x)$ that can change sign, and that can therefore let $''(x)$ change sign, is $2x+1$. Therefore, we find that $f''(x) < 0$ when $2x + 1 < 0$ (i.e. when $x < -1/2$), and that $f''(x) > 0$

when $2x + 1 > 0$ (i.e. when $x > -1/2$). Moreover, we know that at $x = -1/2$ the function has an inflection point.

Now we have all the information to plot the function.

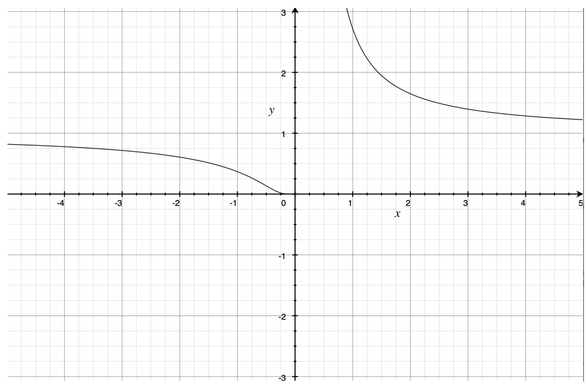


Figure 5.3: Plot of the function $f(x) = e^{1/x}$

5.4 De l'Hôpital's rule

While computing limits we have found several cases where we had to bypass the issue of having an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. We now consider a useful application of derivatives to compute the limits of indeterminate forms of this type.

Method 5.4.1 (**De L'Hôpital's rule**). *Suppose that f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains the point c, or such that c is at one of the extremes of I, except possibly at c where c can be infinite. Then, if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \pm\infty$, i.e. if we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then we have the equality*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$

if the limit on the right hand side exists (finite or infinite).

Example 5.4.2. Consider the limit

$$\lim_{x\to 1} \frac{\ln x}{x-1}.$$

In this case, we have that both numerator and denominator go to zero, meaning that we have a $\frac{0}{0}$ indeterminate form. We can use de l'Hôpital's rule with $f(x) = \ln x$ and $g(x) = x - 1$ to compute the limit. Since $\frac{d}{dx} \ln x = \frac{1}{x}$ and $\frac{d}{dx}(x-1) = 1$, we get

$$\lim_{x\to 1} \frac{f'(x)}{g'(x)} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = 1.$$

So,

$$\lim_{x\to 1} \frac{\ln x}{x-1} = 1.$$

Example 5.4.3. Compute the limit

$$\lim_{x\to\infty} \frac{e^x}{x^2}.$$

This limit is an indeterminate for of type $\frac{\infty}{\infty}$. Apply de l'Hôpital's rule and compute the limit of

$$\lim_{x\to\infty} \frac{e^x}{2x}.$$

This is again an indeterminate form of type $\frac{\infty}{\infty}$, so we can again apply de l'Hôpital's rule and we now have the limit

$$\lim_{x\to\infty} \frac{e^x}{2} = \infty,$$

which now simply gives us that

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \infty.$$

Exercise 5.4.4. Show that for any choice of a positive integer n, we have

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty.$$

Another important type of indeterminate form is of type $0 \cdot \infty$. This happens whenever we have a product of two functions $f(x)$ and $g(x)$ one of which has limit that goes to zero, and the other one that has a limit that goes to $\pm\infty$. This is an indeterminate form because depending on the specific limit, the answer can vary. Consider for instance the following example, showing that various types of answers can be found when having a product of $0 \cdot \infty$.

Example 5.4.5. Take the functions $f(x) = x$ and $g(x) = 1/x^2$. Then,

$$\lim_{x\to\infty} f(x) = \infty,$$

$$\lim_{x\to\infty} g(x) = 0,$$

but obviously we have

$$\lim_{x\to\infty} f(x)g(x) = \lim_{x\to\infty} 1/x = 0.$$

Now define $f(x) = x^2$, and $g(x) = 1/x$. We can see now that

$$\lim_{x\to\infty} f(x) = \infty,$$

$$\lim_{x\to\infty} g(x) = 0,$$

but obviously we have

$$\lim_{x\to\infty} f(x)g(x) = \lim_{x\to\infty} x = \infty.$$

Also, the limit of an infinite limit times a zero limit can also give a constant! Consider for instance the case where $f(x) = x$, and $g(x) = 1/x$. Then we see that

$$\lim_{x\to\infty} f(x) = \infty,$$

$$\lim_{x\to\infty} g(x) = 0,$$

but obviously we have

$$\lim_{x\to\infty} f(x)g(x) = \lim_{x\to\infty} 1 = 1.$$

Exercise 5.4.6. Find examples similar to the previous ones where you display all possible behaviors of the limits of type $0 \cdot \infty$, but using a limit as $x \to a$ for some number a of your choice.

We can deal with an indeterminate form of this type by reducing the problem to a case where we can apply de l'Hôpital's rule! This is done by writing a product as a quotient, using the fact that

$$f(x)g(x) = \frac{f(x)}{\frac{1}{g(x)}},$$

or also

$$f(x)g(x) = \frac{g(x)}{\frac{1}{f(x)}},$$

The following example showcases how to do this.

Example 5.4.7. Consider the limit

$$\lim_{x\to 0^+} x \ln x.$$

Of course this is a $0 \cdot \infty$ type of indeterminate form, since $\ln x$ goes to $-\infty$, as x approaches zero from the right. We can write $x \ln x = \frac{\ln x}{\frac{1}{x}}$, which now gives us an $\frac{\infty}{\infty}$ indeterminate form, to which we can apply de l'Hôpital's rule!

We get

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$

When we have a difference of functions $f(x) - g(x)$, such that both functions go to infinity (or both to negative infinity), then we have a difference of type $\infty - \infty$, where again all sor tof scenarios can happen. In this case, we again try to reduce the limit to an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ to apply de l'Hôpital's rule.

Example 5.4.8. Consider the limit $\lim_{x\to\infty} e^x - x$. Obviously we have a situation of type $\infty - \infty$.

What we can do here, is to create a quotient where we get $\frac{\infty}{\infty}$. We do this by grouping a factor of x. We have:

$$e^x - x = x(\frac{e^x}{x} - 1).$$

Now, we can use de l'Hôpital's rule to see that $\lim_{x\to\infty} \frac{e^x}{x} = \infty$. So, the initial limit gives us

$$\lim_{x\to\infty} e^x - x = \lim_{x\to\infty} x(\frac{e^x}{x} - 1) = \infty,$$

since both terms in the product go to ∞.

Other three types of indeterminate forms are $0^0, \infty^0, 1^\infty$. It is in fact easy to construct examples where each of these forms give different types of behaviors for the limits.

These indeterminate forms arise when we have situations of type $y = f(x)^{g(x)}$, and the method to solve them is by applying either of the following two procedures:

- Take logarithms of the function: $\ln y = g(x) \ln f(x)$.
- Use the fact that exponential and logarithms are one the inverse to the other by writing $y = e^{g(x) \ln f(x)}$.

Example 5.4.9. Compute the limit:

$$\lim_{x\to 0^+} x^x.$$

We have an indeterminate form of type 0^0. Use the second rule given above to write $x^x = e^{x\ln x}$. So, we have to compute

$$\lim_{x\to 0^+} x^x = \lim_{\to 0^+} e^{x\ln x}.$$

From Example 5.4.7 we know that $\lim_{\to 0^+} x\ln x = 0$, so that we have

$$\lim_{x\to 0^+} e^{x\ln x} = e^0 = 1.$$

Example 5.4.10. Compute the limit:

$$\lim_{x\to\infty} (\ln x)^{\frac{1}{x}}.$$

Observe that this is an indeterminate form of type ∞^0. Taking logarithms, we have that

$$\lim_{x\to\infty} \ln((\ln x)^{\frac{1}{x}}) = \lim_{x\to\infty} \frac{1}{x}\ln\ln x.$$

This is now an indeterminate form of type $0\cdot\infty$. We know how to deal with this. Write the function as a fraction: $\frac{1}{x}\ln\ln x = \frac{\ln\ln x}{x}$. Now we can see right away that our limit is an $\frac{\infty}{\infty}$ indeterminate form. Apply de l'Hôpital's rule to this limit.

$$\lim_{x\to\infty} \frac{\ln\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x\ln x}}{1} = \lim_{x\to\infty} \frac{1}{x\ln x} = 0.$$

So, $\lim_{x\to\infty} \frac{1}{x}\ln\ln x = 0$ as well, which gives that $\lim_{x\to\infty} \ln((\ln x)^{\frac{1}{x}}) = 0$. This implies (why?) that

$$\lim_{x\to\infty} (\ln x)^{\frac{1}{x}} = 1.$$

5.5 Optimization

The concepts of minimization and maximization have important applications in real life, where we often encounter the problem of performing some task with the minimum "cost", or "maximum" gain. For instance, one might want to determine the optimal route between two points in order to minimize the distance. Or we might want to minimize production costs and maximize profits of a business.

Optimization problems have recently found fundamental applications in artificial intelligence, especially machine learning, where training neural networks is an alias name for optimizing a neural network function with respect to some objective.

We show a typical simple example of optimization problem.

Example 5.5.1. Suppose that a farmer has 2400 ft of fencing and needs to fence off a rectangular area which borders a (straight) river. We would like to find how the farmer should define the dimensions of the fecning in order to maximize the area enclosed in the fence.

Observe first that since one of the sides of the area will border the river, we need to use fencing only for 3 sides of the rectangle. Moreover, since two sides are the same, the way we can break down the fencing into 3 sides needs to be of the form:

$$2x + y = 2400.$$

From here we can obtain that $y = 2400 - 2x$. Now, having obtained y as a function of x, and recalling that the area of a rectangle is just the product of two adjacent sides, we can write down the area enclosed by the fencing as a function of x (which is one of the sides). We get

$$A(x) = xy = x(2400 - 2x) = 2400x - 2x^2.$$

So, the area is just a polynommial in x. Since we have two sides of size x, and 2400 ft of fence, x can be at most 1200, which would give us a rectangle that is collapsed to a line. So, the values that x gets are between 0 and 1200. We want to find the maximum for $A(x)$ such that $0 \leq x \leq 1200$.

To find the maximum of this function, we now proceed by applying the Closed Interval Method. We need to find the critical points. Since $A(x)$ is just a polynomial, we know that it has no points where derivatives do not exist. Therefore, we just need to obtain where $A'(x) = 0$. We have

$$A'(x) = 2400 - 4x,$$

which gives $x = 600$. So, to obtain the maximum value, we just need to compare $A(x)$ evaluated at the critical point $x = 600$, as well as $A(x)$ evaluated at the extremes of the interval $[0, 1200]$. We find that $A(600) = 720,000$, while $A(0) = A(1200) = 0$. So, to maximize the area, the farmer needs to have a rectangle that has two sides of size $x = 600$, and a size $y = 2400 - 2x = 2400 - 1200 = 1200$.

The next example is similar to the previous one, but it adds a constraint on how the rectangle can vary, making it slightly more complex.

Example 5.5.2. We want to find the rectangle of largest area with the bottom two vertices lying on the parabola $y = x^2$, and with the top side on the line $y = a$.

Here, two points have to be of type $(x, \pm x^2)$, since these are the points of the parabola. Observe that since one side is placed at height equal to a, we have that the vertical sides of the rectangle have size of $a - x^2$. The horizontal side has size $2x$ of course, since the points have horizontal coordinates given by $-x$ and x. Then, the area is given by the formula

$$A(x) = 2x(a - x^2) = 2xa - 2x^3.$$

The values of x are between 0 and $\sqrt{a}$ (why?). Again, to maximize here, we need to use the Closed Interval Method. We know that the function A is differentiable everywhere, so we just need to find the zeros of the derivative of A. We get

$$A'(x) = 2a - 6x^2.$$

The solutions to this equation are $x = \pm\sqrt{\frac{a}{3}}$, which gives us the critical point $\sqrt{\frac{a}{3}}$. We now have to compare all possibilities, so we have to evaluate A at 0, $\sqrt{a}$ and $\sqrt{\frac{a}{3}}$. Since $A(0) = A(\sqrt{a}) = 0$ (of course), then the maximum is reached at $x = \sqrt{\frac{a}{3}}$.

The following method is useful for finding absolute maxima and minima.

Method 5.5.3 (First Derivative Test for Absolute Extrema). Let f be continuous on the interval I, and let c be a critcal number of f. Then, we have the following.

- If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, $f(c)$ is an absolute maximum.
- If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, $f(c)$ is an absolute minimum.

Example 5.5.4. Suppose that we want to find the dimensions of a cylindrical can that is supposed to hold 1L of oil, in a way that we minimize the metal used to produce the can, in order to decrease the production costs.

Minimizing the amount of metal used to produce the can amounts to minimizing the surface of the cylinder. We want to express the area as a function of the raidius of the cylinder. We call r the radius of the discs at the base and top of the cylinder, while h is the height of the cylinder.

Observe that the area consists of two discs, or radius r, and the cylindrical vertical surface that connects them. The area of a disc of radius r is given by πr^2. The area of the vertical sheet is given by the perimeter of the circle times the height h of the cylinder. We get $2\pi rh$. So, the total area is given by

$$A = 2\pi r^2 + 2\pi rh.$$

Now, there are two variables in this formula, namely r and h. We need somehow to reduce the problem to something depending on only one variable. We do this by using the volume of the cylinder. In fact, we have that the volume is given by $\pi r^2 h$,and this is $1L$, which gives us

$$\pi r^2 h = 1\text{L} = 1000\text{cm}^3.$$

So, we get

$$h = \frac{1000}{\pi r^2}.$$

We now can write A as a function only of the variable r, obtaining

$$A(r) = 2\pi r^2 + 2\pi r\frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r},$$

where $r > 0$. Of course, $r = 0$ would be a cylinder with no volume, which is not something possible if we want to have one liter volume. We need to find the critical values of $A(r)$. These are points where $A'(r) = 0$, since A is differentiable for all $r > 0$. We have

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}.$$

We find that $A'(r) = 0$ for $r = \sqrt[3]{\frac{500}{\pi}}$. Now, we can apply Method 5.5.3 to the interval $I = (0, \infty)$ to say that $r = \sqrt[3]{\frac{500}{\pi}}$ is a point of absolute minimum. In fact, it is easy to see that $A'(r) < 0$ for $r < \sqrt[3]{\frac{500}{\pi}}$ amd $A'(r) > 0$ for $r > \sqrt[3]{\frac{500}{\pi}}$.

Now, to complete, we just need to obtain the corresponding value of h, which is done by plugging the value $r = \sqrt[3]{\frac{500}{\pi}}$ in the formula

$$\pi r^2 h = 1\text{L} = 1000\text{cm}^3.$$

relating r and h. We get $h = 2r$.

5.6 Newton's Method

We now describe a method for solving equations numerically, based on differentiation. This is generally an important problem, since in many situations we are required to solve certain equations. However, exact approaches to obtain solutions of equations are very limited. For instance, given a general fifth order polynomial, we do not have a formula that tell us what the solutions are. Newton's method deals with this problem. This is an iterative method that produces approximations for a solution with higher accuracy as iterations increase.

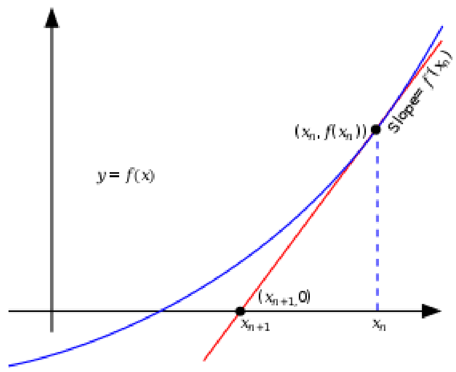


Figure 5.4: Graphic explanation of Newton's method (from Wikipedia)

The method is based on the geometric idea shown in Figure 5.6. Given an function of type $y = f(x)$, for which we want to find a solution to the corresponding equation $f(x) = 0$, we start the method with an approximate guess for a solution. In this case this is called x_n, where n is zero at the initialization, but is higher than zero in the next iterations. The zero is obtained where the blue line crosses the horizontal line corresponding to $y = 0$. To find a better approximation, in this case called x_{n+1} because it represents the next approximation than x_n, we see that if we take the tangent line to the function at the point $(x_n, f(x_n))$ and get the intersection point between this line and $y = 0$, we get closer to a solution. In equations, the tangent line is given by

$$y - f(x_n) = f'(x_n)(x - x_n),$$

since $f'(x_n)$ is the sloper of the tangent (as we know very well from our study of derivatives). When x gets the value x_{n+1} we know that $y = 0$, since x_{n+1} is exactly the point where the tangent line meets the line $y = 0$. Therefore, we have

$$-f(x_n) = f'(x_n)(x_{n+1} - x_n).$$

In the assumption that $f'(x_n)$ is not zero, we can solve for x_{n+1} which gives us

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This procedure gives us a way of producing a better approximation for a solution (x_{n+1}), once we start with some approximate solution (x_n).

As we perform this procedure more and more times, we produce a *sequence* of approimations x_n for $n = 0, 1, \dots$ where n increases. If we get closer and closer to the actual solution, which we call r for "root", then we say that the sequence of approximations converges to a root or solution of the equation, and we write

$$\lim_n x_n = r.$$

We summarize the whole discussion in the folllowing.

Method 5.6.1 (Newton's Method). Let $f(x)$ be a differentiable function, of which we want to find a root, i.e. a number r such that $f(r) = 0$. Let x_0 be a (rough) approximation of a solution to the equation $f(x) = 0$. We can proceed iteratively (in the assumption that at each step the derivative does not vanish) according to the rule

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example 5.6.2. Suppose that we want to find the square root of 2, but that we do not have a calculator at hand. We can then use Newton's method to obtain the value. In fact, $\sqrt{2}$ is that value that satisfies the equation $x^2 - 2 = 0$. This means that we can set $f(x) = x^2 - 2$ and use the method. Here we have $f'(x) = 2x$. Therefore, the iteration takes the form

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}.$$

We need a starting point, i.e. we need some x_0. We will set $x_0 = 1$, which is a relatively accurate guess for the square root of 2. Then we have

- $x_0 = 1$.
- Then $x_1 = x_0 - \frac{x_0^2-2}{2x_0} = 1 - \frac{1-2}{2} = 1 + 1/2 = 1.5$.
- Repeat: $x_2 = x_1 - \frac{x_1^2-2}{2x_1} = 1.5 - \frac{(1.5)^2-2}{2 \cdot 1.5} = 1.41\bar{6}$.
- Now, $x_2^2 = 2.0069$. So, $|2 - x_2^2| < 7 \cdot 10^{-3}$.

We have found a root of 2 with an accuracy of 10^{-3}.

Now a question arises: How do we know when to stop the iterations? There is a useful criterion that tells us when the sequence is converging to he required solution. The criterion is that if we want to achieve a solution with an accuracy of say 10^{-5}, then we can check subsequent iterations x_n and x_{n+1} each time that we produce a new one. In this case, when $|x_{n+1} - x_n| < 10^{-5}$, we can stop the iterations.

Example 5.6.3. Let us now compute a root of the equation $\cos(x) - x = 0$. Here we have for the iterations

$$x_{n+1} = x_n - \frac{\cos(x) - x}{-\sin(x) - 1}.$$

From a simple drawing of the functions $\cos(x)$ and x, we see that the point were they are the same falls below $x = 1$. So, we initialize the iterations with $x_0 = 1$. We get the sequence of subsequent values:

- $x_1 = 0.75036$.
- $x_2 = 0.73911$.
- $x_3 = 0.73908$.

Since $|x_3 - x_2|$ is in the order of 10^{-4}, we expect that we have achieved a relatively good convergence to the solution.

Chapter 6

Integrals

We now consider the notion of integral. As we will see, integration refers to the computation of geometric notions such as the area under a given curve. This is the motivational agument that naturally gives rise to integrals, similarly to how derivatives were associated to velocities and the slope of a tangent line to a curve.

6.1 Antiderivatives and Indefinite Integrals

The problem of finding antiderivatives is somehow the opposite of finding the derivative of a function. In fact, suppose that we know the velocity of a particle as a function of time. We would like to know the function that relate space to time. In fact, we know that the derivative of space with respect to time gives us exactly the velocity. Our problem, therefore, is that of finding a function whose derivative is the given velocity. This is the reason why these objects are called antiderivatives. We formalize this discussion in the following.

Definition 6.1.1. Let f be a function defined on the interval I. A function F is called an *antiderivative* (or sometimes a primitive) of f if $F'(x) = f(x)$. In other words, f is the derivative of F.

Remark 6.1.2. It is clear that antiderivatives are uniquely determined. In fact, observe that if F is an antiderivative of f, then for any number c we have also that $F + c$ is an antiderivative of f (why?).

The following result shows that adding constants to any antiderivative gives us exactly all possible antiderivatives of a function. So, in a sense, the previous remark was as general as possible.

Theorem 6.1.3. *Let F be an antiderivative of f, which is defined on the interval I. Then, if G is another antiderivative of f, there exists a constant c such that*

$$F(x) = G(x) + c.$$

Proof. By definition we have that $F'(x) = G'(x)$ for all x in I. Then, the function $F(x) - G(x)$ has zero derivative on all I. Then, by Theorem 5.2.4 $F(x) - G(x)$ is a constant, let us call this c. It follows that $F(x) = G(x) + c$. □

Definition 6.1.4. Let f be a function defined on I and let F be an antiderivative of f. We define the *indefinite integral* of f as the set of all antiderivatives of f, and indicate it as

$$\int f(x)dx.$$

Observe that by Theorem 6.1.3 $\int f(x)dx$ consists of all the functions $F(x) + c$, where c is an arbitrary number (a constant).

Example 6.1.5. Obtain $\int (x^2 + x)dx$.

We need to find a function whose derivative gives x^2+x. We know that derivatives of polynomials give polynomials. Also, we know the expression for it: $\frac{d}{dx}(a_n x^n + \cdots a_1 x + a_0) = n a_n x^{n-1} + \cdots a_1$. We apply this here. We take $F(x) = \frac{1}{3}x3 + \frac{1}{2}x^2$. Computing the derivative of F we get

$$F'(x) = \frac{1}{3}3x^2 + \frac{1}{2}2x = x^2 + x = f(x).$$

Now, $\int f(x)dx$ is the set of functions of type $\frac{1}{3}x3 + \frac{1}{2}x^2 + c$, where c is a constant.

Exercise 6.1.6. Can you find a formula for the indefinite integral of $a_n x^n + \cdots + a_1 x + a_0$?

Example 6.1.7. Find the indefinite integral of $f(x) = \frac{1}{x}$ over an interval of positive numbers.

We know that the derivative of the function $\ln x$ is exactly $\frac{1}{x}$. Then we have that $\int f(x)dx$ is the set of functions of type $\ln x + c$ where c is a constant.

It is easy to see that if the interval is contained in the negative numbers, then we just have to replace $\ln x$ with $\ln -x$.

Example 6.1.8. Find a function $F(x)$ with the following properties:

- $F'(x) = \sin x$.
- $F(0) = 5$.

We first need to find the indefinite integral of $\sin x$, since $F'(x) = \sin x$ means that F has to be an antiderivative of $\sin x$. We know that $\frac{d}{dx}\cos x = -\sin x$, therefore an antiderivative of $\sin x$ is given by $-\cos x$. The indefinite integral consists of functions of type $F(x) = -\cos x + c$. Now imposing the condition that $F(0) = 5$ we get $F(0) = -1 + c = 5$, which implies $c = 6$. So, the function we are looking for is $F(x) = -\cos x + 6$.

We have the following rules for finding particular antiderivatives.

- For $f(x) = x^n$, we have $F(x) = \frac{x^{n+1}}{n+1}$.
- For $f(x) = \frac{1}{x}$, $F(x) = \ln x$. For an interval in the negative numbers we have $F(x) = \ln -x$.
- For $f(x) = e^x$, we have $F(x) = e^x$.
- For $f(x) = \cos x$, $F(x) = \sin x$.
- For $f(x) = \sin x$, $F(x) = -\cos x$.
- For $f(x) = \sec^2 x$, $F(x) = \tan x$.

- For $f(x) = \sec x \tan x$, $F(x) = \sec x$.
- For $f(x) = \frac{1}{\sqrt{1-x^2}}$, $F(x) = \sin^{-1} x$.
- For $f(x) = \frac{1}{1+x^2}$, $F(x) = \tan^{-1} x$.
- For $f(x) = \cosh x$, $F(x) = \sinh x$.
- For $f(x) = \sinh x$, $F(x) = \cosh x$.

6.2 The area under a curve

Suppose we are given a curve, and assume that we are interested in finding the area between the curve and the $\vec{x}$-axis. We know how to compute the area of polygons from elementary geometry, but it is not clear how to generalize the notion of area to the case of curved objects. The intuition behind how to proceed is that we can approximate a curved object by using polygons, e.g. rectangles, and that we can make this approximation increasingly accurate by using more and more rectangles. This situation is depicted in Figure 6.2 and Figure 6.2. Using just two rectangles we obtain a very rough approximations of the area, while increasing the number of rectangles we get a better estimate of the area.

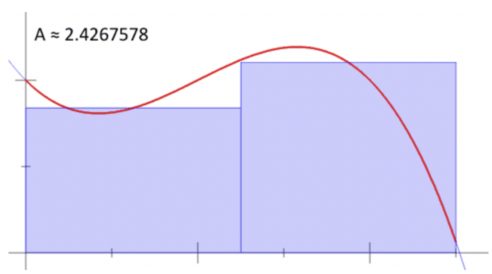


Figure 6.1: A rough approximation of the area under a curve using rectangles (from Wikipedia)

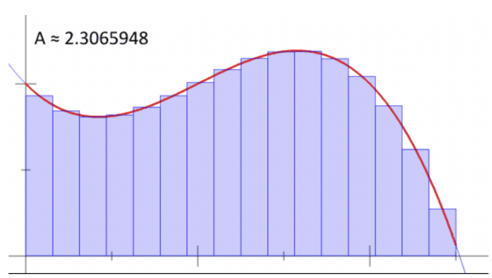


Figure 6.2: A good approximation of the area under a curve using rectangles (from Wikipedia)

The way to formalize this procedure is the following. Assume that the curve is defined through the function $f(x)$ defined on the interval $[a, b]$. Then, we introduce a partition of the interval $[a, b]$. This means that we find points $a = x_0, x_1, \cdots, x_n = b$ subdividing $[a, b]$ into subintervals. We assume that the distance between x_i and x_{i+1} is the same, so that each subinterval has the same size Δx. We now evaluate the function f on the points $x_1, \ldots, x_n$ (we skip x_0, can you see why?). Now, we can define the rectangles $R_1, R_2, \ldots, R_n$ where $R_1 = f(x_1)\Delta x$, $R_2 = f(x_2)\Delta x$ and so on up to $R_n = f(x_n)\Delta x$.

As observed above, as the number of rectangles increases, i.e. as n grows to ∞, we approximate the area under the curve $y = f(x)$ with more accuracy. Therefore, we pose the following (rather informal) definition.

Definition 6.2.1. Given $f(x)$ defined over $[a, b]$, the area under the curve $y = f(x)$ is the limit of the areas of the rectangles constructed above as n goes to ∞. In symbols, we can write

$$A = \lim_{n\to\infty} R_1 + \cdots + R_n = \lim_{n\to\infty} f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

Similarly, we can also construct the rectangles R_i above by evaluating $f(x)$ at the points $x_0, x_1, \ldots, x_{n-1}$. We get rectangles $L_1 = f(x_0)\Delta x$, $L_2 = f(x_1)\Delta x$ and so on up to $L_n = f(x_{n-1})\Delta x$.

When f is a continuous function, it turns out that the limit in Definition 6.2.1 always exists and also the limit

$$\lim_{n\to\infty} L_1 + \cdots + L_n = \lim_{n\to\infty} f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$$

exists and the two limits are the same. In fact, even more is true. One could choose points x_i^* arbitrarily in each interval $[x_i, x_{i+1}]$ for each $i = 0, 1, \ldots, n-1$ and construct the limit

$$\lim_{n\to\infty} f(x_0^*)\Delta x + f(x_1^*)\Delta x + \cdots + f(x_{n-1}^*)\Delta x$$

and this limit would be the same as the two limits above. For instance, in Figure 6.2 and Figure 6.2 the middle point $x_i^* = \frac{x_i + x_{i+1}}{2}$ of each interval $[x_i, x_{i+1}]$ has been chosen to construct the rectangles.

6.3 Definite Integrals

Definition 6.3.1. Let f be a function defined over the interval $[a, b]$, which we subdivide in n parts. Then, if the limit

$$\lim_{n\to\infty} f(x_0^*)\Delta x + f(x_1^*)\Delta x + \cdots + f(x_{n-1}^*)\Delta x$$

is independent of the chosen points x_i^*, we will call it the *definite integral of f from a to b*, and write

$$\int_a^b f(x)dx = \lim_{n\to\infty} f(x_0^*)\Delta x + f(x_1^*)\Delta x + \cdots + f(x_{n-1}^*)\Delta x.$$

In this case, we say that the function is intebrable on $[a, b]$.

Remark 6.3.2. The definition given above is also called the *Riemann integral*, as opposed to other types of integration (e.g. Lebesgue) which we do not consider in this course. The sums of type $R_1 + \cdots + R_n$ and so on (for arbitrary choices of sample points x_i^*) are called Riemann sums.

Theorem 6.3.3. *If f is a continuous function everywhere on $[a, b]$ except possibly at a finite number of points where it has a jump discontinuity, then f is integrable over $[a, b]$, i.e. $\int_a^b f(x)dx$ exists.*

Let us now compute an integral, making use of Theorem 6.3.3.

Example 6.3.4. Let $f(x) = x^2$ be defined over the interval $[0, 1]$. We want to compute $\int_0^2 x^2 x$.

First, observe that $f(x) = x^2$ is continuous, and therefore applying Theorem 6.3.3 we know that it is integrable over $[0, 1]$. Moreover, we know that however we choose to sample points x_i^* in the subintervals, the limit will be the same (equal to the integral). So, we will compute Riemann sums of type $R_1 + \cdots + R_n$, which means that our sample points will be the rightmost points of the subintervals.

Divide $[0, 1]$ in n parts. This means that $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. Also, the points will be $x_0 = 0$, $x_1 = 0 + \Delta x = \frac{1}{n}$, $x_2 = x_1 + \Delta x = \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$ and so on. So, point $x_i = \frac{i}{n}$ up to $x_n = \frac{n}{n} = 1$. The function f evaluated at the points x_i gives us $f(x_i) = f(\frac{i}{n}) = \frac{i^2}{n^2}$.

Then, $R_1 = f(x_1)\Delta x = \frac{1^2}{n^2}\frac{1}{n} = \frac{1}{n^3}$, and more generally for each i we have $R_i = \frac{i^2}{n^2}\frac{1}{n} = \frac{i^2}{n^3}$. Therefore, we get

$$\begin{aligned} R_1 + \cdots + R_n &= \frac{1}{n}f(\frac{1}{n}) + \frac{1}{n}f(\frac{2}{n}) + \cdots + \frac{1}{n}f(\frac{n}{n}) \\ &= \frac{1}{n}(\frac{1}{n})^2 + \cdots + \frac{1}{n}(\frac{n}{n})^2 \\ &= \frac{1}{n}\frac{1}{n^2}(1^2 + 2^2 + \cdots + n^2). \end{aligned}$$

We now use the following identity (whose proof we omit):

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Then, we get

$$\sum_{i=1}^{n} R_i = \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}.$$

To take the limit of this quantity (which is the integral by definition), we will treat this quantity as we have done for the limits with respect to x that goes to ∞. We have

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} R_i &= \lim_{n\to\infty} \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \\ &= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \\ &= \lim_{n\to\infty} \frac{2 + 3/n + 1/n^2}{6} \\ &= 2/6 \\ &= 1/3. \end{aligned}$$

Remark 6.3.5. When we want to compute approximations of integrals, it is often useful to choose as sample point x_i^* the midpoint of the subinterval $[x_i, x_{i+1}]$. Usually, this gives us a good approximation of the integral.

6.4 Properties of Integrals

Our definition of integral used the fact that $a < b$. When we have instead $a > b$ we set

$$\int_a^b f(x)dx = -\int_a^b f(x)dx.$$

Also, if $a = b$ we set

$$\int_a^a f(x)dx = 0.$$

Integrals satisfy the following properties.

Proposition 6.4.1. *The following properties hold.*

1. $\int_a^b kdx = k(b-a)$ *for any constant* k.
2. $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
3. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.

Proof. The proof of these facts is simple, and it is left to the reader as an exercise. □

Theorem 6.4.2. *The following results hold.*

- $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$ *for any choice of* b *between* a *and* c.
- *If* $f(x) \geq 0$ *in* $[a,b]$, *then* $\int_a^b f(x)dx \geq 0$.
- *If* $f(x) \geq g(x)$ *in* $[a,b]$, *then* $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
- *If* $m \leq f(x) \leq M$ *in* $[a,b]$, *then* $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

6.5 The Fundamental Theorem of Calculus

The fundamental theorem of calculus (FTC for short) is the cornerstone of Calculus I, as the name suggests. We will prove it in two parts. The first part shows that if we have a continuous function $f(x)$ defined over an interval $[a,b]$, then there exist antiderivatives to $f(x)$, and it also shows how to construct such antiderivatives (we already know that knowing a particular one gives us all of them by translations by a constant).

Theorem 6.5.1 (FTC part I). *Let* $f(x)$ *be continuous in* $[a,b]$. *Then, there exists an antiderivative* $F(x)$ *of* $f(x)$, *given by*

$$F(x) = \int_a^x f(t)dt.$$

The function F *is continuous on* $[a,b]$ *and differentiable in* (a,b), *with* $F'(x) = f(x)$.

Proof. We show that for any x in (a, b) F is differentiable and $F'(x) = f(x)$. For $x = a$ or $x = b$ we can proceed in the same way to show continuity in a one-sided manner, i.e. taking right and left limits.

We compute $F(x + h) - F(x)$ when $h > 0$. We have

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \int_a^x f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \int_x^{x+h} f(t)dt, \end{aligned}$$

where in the second equality we have used the first property in Theorem 6.4.2. Therefore, the quotient which we need to compute the derivative of F at x is given by

$$\frac{F(x+h) - F(x)}{h} = \frac{\int_x^{x+h} f(t)dt}{h}. \tag{6.1}$$

Now, since f is continuous in the interval $[x, x + h]$, it follows that it has a minimum m and a maximum M, due to the Extreme Value Theorem. Using the fourth property of Theorem 6.4.2 we get

$$m(x + h - x) \leq \int_x^{x+h} f(t)dt \leq M(x + h - x),$$

and therefore

$$mh \leq \int_x^{x+h} f(t)dt \leq Mh. \tag{6.2}$$

By dividing Inequality (6.2) by h we find

$$m \leq \frac{\int_x^{x+h} f(t)dt}{h} \leq M. \tag{6.3}$$

By Equation (6.1) we get

$$m \leq \frac{F(x+h) - F(x)}{h} \leq M. \tag{6.4}$$

Since m and M are in $[f(x), f(x + h)]$, when we take the limit $h \to 0$ the interval $[f(x), f(x + h)]$ collapses to the point $f(x)$, because f is continuous, forcing m and M to be equal to $f(x)$. Therefore, by the Squeeze Theorem,

$$\lim_{h\to 0} m = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} M,$$

which is equal to $f(x)$. This shows that $F'(x) = f(x)$.

To deal with the case where $h < 0$ we can proceed in the same way by being careful about the signs. For the cases $x = a$ and $x = b$, as stated above, the proof is identical but only one sided limits are taken. □

Theorem 6.5.2 (FTC part II). *If $f(x)$ is continuous on $[a, b]$, then*

$$\int_a^b f(x)dx = F(b) - F(a),$$

where F is any antiderivative of f, i.e. where $F'(x) = f(x)$.

Proof. Consider the antiderivative $F(x) = \int_a^x f(t)dt$, which was defined in part I of FTC. We also know that $F(a) = \int_a^a f(t)dt = 0$. So, it follows that $F(b) - F(a) = F(b) = \int_a^b f(t)dt$, which can be written simply as $F(b) - F(a) = \int_a^b f(x)dx$, simply by renaming the variable t by x. Now, we have to show that the same holds for any antiderivative. We have shown that if f is continuous, any two antidereivatives differ by a constant for all x in (a, b). By continuity it also holds that F and any other antiderivative G differ by a constant k also on a and b. Therefore, $G(b) - G(a) = [F(b) + k] - [F(a) + k] = \int_a^b f(x)dx$, which completes the proof. □

Theorem 6.5.2 is very useful to compute integrals. For instance, computing the integral of $f(x) = x^2$ in $[0, 1]$ becomes much easier, compared to the direct method used in Example 6.3.4.

Example 6.5.3. We compute the integral $\int_0^1 x^2 dx$. An antiderivative of $f(x) = x^2$ is given by $F(x) = \frac{x^3}{3}$. Therefore, using Theorem 6.5.2 we have that

$$\int_0^1 x^2 dx = F(1) - F(0) = \frac{1^2}{3} - \frac{0^2}{3} = \frac{1}{3}.$$

Observe that this agrees with the result found in Example 6.3.4.

6.6 Substitution Rule

We have seen that in order to compute definite integrals, we need to be able to compute antiderivatives. This means that we need to compute the indefinite integrals. We now consider an important technique that can be considered as a sort of inverse to the chain rule.

Method 6.6.1 (Substitution Rule). *Let $f(x)$ be a continuous function defined on the interval I, and let $g(x)$ be a differentiable function with values in I, i.e. such that we can compose the two functions. Then, we have the following equality:*

$$\int f(g(x))g'(x)dx = \int f(u)du, \tag{6.5}$$

where we set $g(x) = u$.

Proof. Suppose that $F(x)$ is an antiderivative of $f(x)$. Then, by the chain rule we have $\frac{d}{dx}(F(g(x))) = F'(g(x))g'(x) = f(g(x))g'(x)$. Therefore, $F(g(x)) + k$ is the most general antiderivative of $f(g(x))g'(x)$. So,

$$\int f(g(x))g'(x)dx = F(g(x)) + k.$$

Now, if we set $g(x) = u$, we also find that

$$\int f(u)du = F(u) + k = F(g(x)) + k.$$

This completes the proof. □

Remark 6.6.2. Observe that the differential du is obtained, informally, as $du = g'(x)dx$. This is motivated by the fact that $\frac{du}{dx} = g'(x)$, which can be multiplied on both sides by dx.

Example 6.6.3. We now compute $\int x^3 \cos(x^4 - 3)dx$.

Observe that the derivative of the argument of the cosine is given by $\frac{d}{dx}(x^4 - 3) = 4x^3$. Therefore, in order to have the derivative of $g(x) = x^4 - 4$ in the integral, we just need a factor of 4. We can obtain this by multiplying and dividing by 4. We have

$$\int x^3 \cos(x^4 - 3)dx = \frac{1}{4}\int \cos(x^4 - 3)4x^3 dx.$$

Now, we can use the substitution rule with $f(u) = \cos(u)$, having set $u = x^4 - 3$. We have

$$\int x^3 \cos(x^4 - 3)dx = \frac{1}{4}\int \cos(u)du = \frac{1}{4}\sin(u) + k = \frac{1}{4}\sin(x^4 - 3) + k.$$

Example 6.6.4. Compute $\int \frac{x}{\sqrt{1+x^2}}dx$.

Here again, we want to solve this integral by using the substitution rule. We have to identify a component that can play the role of $f(u)$, and a component that can play the role of $g(x) = u$. Consider the function $f(g(x)) = \frac{1}{\sqrt{1+x^2}}$, where $f(x) = \frac{1}{\sqrt{x}}$ and $g(x) = 1 + x^2$. We have that $g'(x) = 2x$, so that $\frac{x}{\sqrt{1+x^2}}$ can almost be written as $f(g(x))g'(x)$, where "almost" is due to the fact that we are missing a factor of 2. Once again, we can introduce it in order to be able to apply the substitution rule as in the previous example.

$$\int \frac{x}{\sqrt{1+x^2}}dx = \frac{1}{2}\int \frac{2x}{\sqrt{1+x^2}}dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}du = \sqrt{u} + k.$$

Since $u = g(x) = 1 + x^2$, we get

$$\int \frac{x}{\sqrt{1+x^2}}dx = \sqrt{1+x^2} + k.$$

Comnbining the substitution rule and the Fundamental Theorem of Caclulus part II, we get the rule for definite integration using the substitution rule. Under the same hypothesis as Method 6.6.1, we have

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

The proof is simple, and lef to the reader as an exercise.

Chapter 7

Applications of Integration

7.1 Areas between curves

We have introduced integration as a formalization of the problem of finding the area under a curve. The problem was to find the area between the horizontal x axis, and a given curve $y = f(x)$ defined through some function (which was assumed to be continuous for simplicity). Observe that the horizontal axis is given by the equation $y = 0$. Therefore, we can think of our original problem as finding the area between the function $y = f(x)$ and the function $y = g(x)$ where $g(x) = 0$.

In fact, the same reasoning can be applied for any function $g(x)$. We can therefore define the area between two curves $y = f(x)$ and $y = g(x)$ (in the assumption that $f(x) \geq g(x)$) by the integral

$$A = \int_a^b [f(x) - g(x)]dx.$$

When $f(x)$ and $g(x)$ do not satisfy an equality of type $f(x) \geq g(x)$, then we can use absolute values, to compute the area between $f(x)$ and $g(x)$:

$$A = \int_a^b |f(x) - g(x)|dx.$$

Example 7.1.1. Compute the area enclosed between the parabolas $y = x^2$ and $y = 2x - x^2$, where x varies between 0 and 1.

Using our definition of area between curves, we need to compute

$$\int_0^1 |x^2 - (2x - x^2)|dx = \int_0^1 |2x^2 - 2x|dx = \int_0^1 [2x - 2x^2]dx$$

From the Fundamental Theorem of Calculus (part II), we need to first find an antiderivative of $-2x^2 + 2x$. This is easily done, since the function is a polynomimal. We see that an antiderivative is given by $F(x) = -\frac{2x^3}{3} + x^2$. Then we need to evaluate F at 0 and 1. We have

$$A = \int_0^1 |2x^2 - 2x|dx = F(1) - F(0) = -\frac{2}{3} + 1 = \frac{1}{3}.$$

Remark 7.1.2. We now start using the notation $[F(x)]_a^b = F(b) - F(a)$.

Example 7.1.3. Consider the functions $f(x) = \frac{x}{\sqrt{x^2+1}}$ and $g(x) = x^4 - x$. We want to find the area enclosed between the functions.

We need to understand the extremes of integration. They correspond to where the functions equal each others. By plotting the functions (try it!) and studying their derivatives and concavities, horizontal asymptotes and so on, we can understand that there are two points where $f(x) = g(x)$. One point is rather obvious, and it is $x = 0$. The other one is quite difficult to understand. It is found around $x = 1$. We then apply Newton's Method to find an approximate solution to the equation $f(x) = g(x)$, which is the same as trying to find $f(x) - g(x) = \frac{x}{\sqrt{x^2+1}} - x^4 + x = 0$. Initializing at $x = 1$ and performing two iteraitons we find the approximate solution $x = 1.1909$. So, we need to integrate between 0 and 1.19.

We have

$$A = \int_0^{1.19} [f(x) - g(x)]dx = \int_0^{1.19} f(x)dx - \int_0^{1.19} g(x)dx = \int_0^{1.19} \frac{x}{\sqrt{x^2+1}}dx - \int_0^{1.19} (x^4 - x)dx.$$

An antiderivative of $x^4 - x$ is just given by $\frac{x^5}{5} - \frac{x^2}{2}$, so the second integral is easily solved. To solve the first one, one can use substitution to find the antiderivative $\sqrt{u}$, where $u = x^2 + 1$. Then, indicating by we have

$$A = \int_0^{1.19} \frac{x}{\sqrt{x^2+1}}dx - \int_0^{1.19} (x^4 - x)dx = [u]_1^{1.19^2+1} - [\frac{x^5}{5} - \frac{x^2}{2}]_0^{1.19}.$$

We therefore find $A \approx 0.785$.

7.2 Volumes

To compute volumes of a solid in three-dimensional space, one can proceed similarly to the case of the area under a curve, but where we subdivide the along length of the solid with cross sections. If we derive a formula for the variation of the area of cross sections along the length of the solid, then we can integrate with respect to the length and obtain the full volume. We therefore have the following definition.

Definition 7.2.1. Let S be a solid that lies between $x = a$ and $x = b$. Let $A(x)$ denote the area of a cross section of S at the point x, as a function of x. Here A is assumed to be a continuous function. Then the volume of S is

$$\lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx.$$

Example 7.2.2. Let us compute the volume V of a sphere S of radius r.

To do this, we need to obtain a formula for the area of a cross section of S as the length along the x axis varies, and then we have to integrate this formula between $-r$ and r. In fact, due to symmetry of the sphere, we can simply compute the integral between 0 and r, and then multiply the result by 2.

To compute the area of a cross section at x, i.e. $A(x)$, observe that cross sections of a sphere are just disks. However, the radius of the disk changes depending on x. In other words, it is a function of x. If y is the point on the sphere with x coordinate, we have that

$$A(x) = \pi y^2 = \pi(r^2 - x^2).$$

We now compute the integral between 0 and r of $A(x)$ (and multiply by 2):

$$\begin{aligned} V &= 2\int_0^r A(x)dx \\ &= 2\int_0^r \pi(r^2 - x^2)dx \\ &= 2[r^2x - \frac{x^3}{3}]_0^r \\ &= 2\pi(r^3 - \frac{r^3}{3}) \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

When we have a solid of revolution, i.e. a solid S obtained by rotating a curve about an axis, we can obtain the volume by computing the area undeer the curve and then obtaining a rotation by 2π of it. We get the definition

$$V = \pi \int_0^1 f(x)^2 dx.$$

Example 7.2.3. Consider the function $y = x^3$ and consider the solid S obtained by rotating it around the $\vec{y}$-axis, bounded by $y = 8$.

Since we are rotating about the $\vec{y}$-axis, we need to turn the function from $y = x^3$ to a function of x with respect to y. This is done by writing $x = \sqrt[3]{y}$. We get

$$V = \pi \int_0^8 A(y)dy = \pi \int_0^8 y^{\frac{2}{3}} dy = \pi[\frac{3}{5}y^{\frac{5}{3}}]_0^8 = \frac{96\pi}{5}.$$

7.3 Cylindrical Shells

Consider the problem of computing the volume of a cylindrical shell of thickness Δr and height h. We can write the shell as the difference of two cylinders, one with radius r_1 and the other with radius r_2, having thikness $\Delta r = r_1 - r_2$. To compute the volume in this case then we can subtract the volumes of the two cylinders. We have

$$\begin{aligned} V &= V_1 - V_2 \\ &= \pi r_1^2 h - \pi r_2^2 h \\ &= \pi h(r_1^- r_2^2) \\ &= \pi h(r_1 + r_2(r_1 - r_2) \\ &= 2\pi h \frac{r_1 + r_2}{2} \Delta r. \end{aligned}$$

So, we have written the volume V of the shell as 2π multiplied by height h, and average radius of the shell, which is $\frac{r_1+r_2}{2}$, multiplied the radius difference Δr.

Now, when S is a solid obtained through the rotation of a function $f(x)$ about the $\vec{y}$-axis, we can approximate the volume by several cylindrical shells, whose volume we know to compute, following the previous reasoning. This approximation takes the form

$$V \approx \sum_i V_i = \sum_i 2\pi x_i^* f(x_i^*) \Delta x.$$

We can therefore define the volume of the solid of rotation S by using the formula that turns the previous approximation into an integral:

$$V = 2\pi \int_a^b xf(x)dx. \tag{7.1}$$

As a matter of fact, this does not need to be taken as a definition, but it can actually be proved.

Example 7.3.1. Consider the solid obtained by rotating the region between the function $y = 2x^2 - x^3$ and $y = 0$ about the $\vec{y}$-axis. We want to compute the volume of it

To do so, we use Equation (7.1). To apply the equation we need to find the interval of integration. This done by observing that $2x^2 - x^3 = 0$ at $x = 0$ and $x = 2$.

We have

$$\begin{aligned} V &= 2\pi \int_0^2 xf(x)dx \\ &= 2\pi \int_0^2 x(2x^2 - x^3)dx \\ &= 2\pi \int_0^2 [2x^3 - x^4]dx \\ &= 2\pi[\frac{1}{2}x^4 - \frac{1}{5}x^5]_0^2 \\ &= \frac{16}{5}\pi. \end{aligned}$$

7.4 Work

According to Newton's second law of motion, the force applied to an object to move it along a straight line is given by the formula

$$F = ma = m\frac{d^2s}{dt^2}, \tag{7.2}$$

where m is the mass of the object, $a(t)$ represents the acceleration and $s(t)$ the space. The measure of unit of force, is given by mass times space per squared time. In the International System of Units, this is given by kilograms (kg) times meters (m) per seconds (s) squared. The force is measured in newtons, N, and we have $N = \mathrm{kg} \cdot \mathrm{m/s}^2$. If the acceleration of the particle is constant, then the work done is defined to be the product of force times space:

$$W = Fd.$$

Work is measured in joules J, and it corresponds to netwons times meters: $J = N \cdot m$.

One natural question arises: How do we define the notion of work when the acceleration (i.e. for nonrelativistic speeds the force) is not constant? This is defined as an integral:

$$W = \int_a^b f(x)dx,$$

where $f(x)$ is the force acting on the object at the space point x.

Example 7.4.1. Assume that a particle is acted upon by a force depending on the particle's distance from the origin with equation $f(x) = x^2 + 2x$, where x indicates distance from the origin. How much work is done to move the particle from $x = 1$ to $x = 3$?

We need to integrate the function $f(x)$ from 1 to 3. We have

$$W = \int_1^3 f(x)dx = \int_1^3 (x^2 + 2x)dx = [\frac{x^3}{3} + x^2]_1^3 = 18 - 4/3\ J.$$

Example 7.4.2. Hooke's Law states that the force acting on a loose end of a spring subject to a stretch of x units is proportional to x:

$$f(x) = kx,$$

where k is a constant depending only on the spring.

Suppose that we know that the force on the loose spring which is stretched by 5 cm is $10N$. Assume that the lenght of rest of the spring is 10 cm. What is the work done to stretch the spring from 10 cm to 18 cm?

Since the force for a stretch of 5 is given by $f(x) = kx$, we have $10N = k \cdot 5$ cm, from which we get $k = 2N/\text{cm}$. Now we can compute the work. To stretch the spring from 10 cm to 18 cm, we need to produce a stretch of 0 to 8 cm. This gives us an integral from 0 to 8. The force $f(x) = 2x$, since $k = 2N/\text{cm}$. We have

$$W = \int_0^8 f(x)dx = 2\int_0^8 x = 2[x^2/2]_0^8 = 64\ J.$$

7.5 Mean Value Theorem for Integrals

The average of a discrete number of points is obtained by summing all the points together and dividing by the total number of points. For values $y_1, \ldots, y_n$ we have

$$\bar{y} = \frac{y_1 + \cdots + y_n}{n},$$

where $\bar{y}$ indicates the average.

Suppose now that we have a function $f(x)$ defined over the interval $[a, b]$. We want to compute the average of the n points $f(x_1), \ldots, f(x_n)$ where x_i are sampled in the interval $[a, b]$ with distance between two consecutive points given by Δx. The number Δx satisfies

$$\Delta_x = \frac{b-a}{n}.$$

So, the average is given by

$$\bar{f}(x) = \frac{f(x_1) + \cdots + f(x_n)}{n} = \frac{f(x_1) + \cdots + f(x_n)}{\frac{b-a}{\Delta x}} = \frac{f(x_1) + \cdots + f(x_n)}{b-a}\Delta x.$$

This is a Riemann sum!

Therefore, it makes sense to define the average of the function f on the interval $[a, b]$ as

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx.$$

The following result is called the Mean Value Theorem for Integrals.

Theorem 7.5.1. *Let f be continuous on $[a, b]$. Then there exists a number c in $[a, b]$ such that*

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Proof. Define the function $F(x) = \int_a^x f(t)dt$. From Theorem 6.5.1, we know that F is differentiable and that $F'(x) = f(x)$. From the Mean Value Theorem we have that there exists a point c in $[a, b]$ such that

$$F'(c) = \frac{F(b) - F(a)}{b-a}.$$

Now, $F'(c) = f(c)$, $F(a) = \int_a^a f(x)dx = 0$ and $F(b) = \int_a^b f(x)dx$. This completes the proof. □

Chapter 8

Techniques of Integration

The only approach to integration that we have considered so far is substitution. At this point, we can integrate functions only in some special cases. Namely, either when we are given a function whose antiderivative is known to us (e.g. $f(x) = \sin(x)$), or when we can recognize a multiplication by a factor whose antiderivative is known to us. In the latter case, we can somehow "reverse" the chain rule and obtain an antiderivative. This is the substitution rule. Obviously, it is not very common to fall in either of these situations. We will develop further approaches to integration that combined with the previous methods will allow us to integrate complicated functions.

8.1 Integration by Parts

While substitution can be considered as the inverse of the chain rule, integration by parts is a converse to the Leibniz rule (the product rule).

Method 8.1.1 (Integration by Parts). *Let f and g be two continuously differentiable functions. Then the following equality holds*

$$\int [f(x)g'(x)]dx = f(x)g(x) - \int [f'(x)g(x)]dx. \tag{8.1}$$

For definite integrals, the previous equation gives

$$\int_a^b [f(x)g'(x)]dx = f(b)g(b) - f(a)g(a) - \int_a^b [f'(x)g(x)]dx. \tag{8.2}$$

Proof. The product rule for f and g gives

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x). \tag{8.3}$$

Taking the integral of boths sides of Equation (8.3) gives

$$f(x)g(x) = \int [f'(x)g(x)]dx + \int [f(x)g'(x)]dx, \tag{8.4}$$

where we used the fact that the integral of a derivative is just equal to the function that is being integrated, so that the integral and the derivative sybols "cancel" each other. To be precise, one should also include an integration constant, but since on the right hand side we have two more integrals (which have integration constants as well), this can be absorbed in the remaining integrals. Rearranging terms in Equation (8.4) we obtain

$$\int [f(x)g'(x)]dx = f(x)g(x) - \int [f'(x)g(x)]dx,$$

which completes the proof of the first part. The second part follows from applying Equation (8.1) and the Fundamental Theorem of Calculus (part 2). □

Integration by parts allows us to integrate functions that until now were difficult for us to integrate (in a systematic way).

Example 8.1.2. Consider the integral $\int xe^x dx$. We want to use integration by parts to solve this integral, since substitution is not really helpful here.

Let us set $g'(x) = e^x$ and $f(x) = x$ in Equation (8.1). This choice is motivated by the fact that we know how to "undo" the derivative $g'(x)$, so we know how to get $g(x)$, which is $g(x) = e^x$. Also, when we differentiate $f(x)$, we will get something much simpler: $f'(x) = 1$. Equation (8.1) gives us

$$\int xe^x dx = xe^x - \int 1 \cdot e^x dx.$$

The last integral now is simple to solve: $\int e^x dx = e^x + c$. So, we have found

$$\int xe^x = xe^x - e^x + c.$$

Example 8.1.3. We want to compute $\int \ln x \, dx$. Observe that while we do know how to differentiate $\ln x$, we do not really know any function whose derivative is $\ln x$.

Here we apparently only have one function, so that it seems that applying integration by parts is not possible. Of course, we can think of having a 1 multiplying $\ln x$, and therefore, we can write the integral as

$$\int 1 \cdot \ln x \, dx.$$

Here we can think of $g'(x) = 1$ and $f(x) = \ln x$, since we know how to undo the derivative $g'(x) = 1$, which just gives us a $g(x) = x$. Applying Equation (8.1) we have

$$\int 1 \cdot \ln x \, dx = x \ln x - \int x \cdot (\frac{d}{dx} \ln x) dx.$$

Since $\frac{d}{dx} \ln x = \frac{1}{x}$, we get

$$\int \ln x \, dx = x \ln x - \int 1 dx = x \ln x - x + c.$$

The following is a very interesting example where integration by parts does not seem to be applicable or helpful. We will see how to bypass the issue.

Example 8.1.4. We want to compute now the integral $\int e^x \sin x \, dx$. Here, both functions e^x and $\sin x$ could be considered as the differentiated function $g'(x)$. However, the problems seems to be the fact that once you choose either of them to be $g'(x)$, and the other one to be $f(x)$, taking the derivative $f'(x)$ does not seem to result in a solvable integral. Let us see how to deal with this problem.

First, we just set $f(x) = e^x$, and $g'(x) = \sin x$. Then, we apply Equation (8.1) to get

$$\int e^x \sin x \; dx = -e^x \cos x + \int e^x \cos x \; dx. \tag{8.5}$$

As expected, the right hand side of the previous equation is not any simpler than the integral we started with. We now apply integration by parts again to $\int e^x \cos x \; dx$ and see what happens. Again, we set $f(x) = e^x$ and $g'(x) = \cos x$. We obtain

$$\int e^x \cos x \; dx = e^x \sin x - \int e^x \sin x \; dx. \tag{8.6}$$

Substituting Equation (8.6) in Equation (8.5) we find

$$\int e^x \sin x \; dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \; dx,$$

which can be written upon rearranging terms as

$$2 \int e^x \sin x \; dx = -e^x \cos x + e^x \sin x.$$

Finally, this gives us

$$\int e^x \sin x \; dx = 1/2(-e^x \cos x + e^x \sin x) + c,$$

where we have added the integration constant in the last step.

Therefore, applying integration by parts twice, we have been able to integrate the function $e^x \sin x$.

8.2 Trigonometric integrals and trigonometric substitutions

We have already seen how to use the rules of differentiation of trigonometric functions to obtain some direct integrals. However, we generally need to use a combination of known trigonometric integrals, integration by parts and by substitution to be able to integrate more complex trigonometric functions. We begin with an interesting example.

Example 8.2.1. Consider the integral $\int \cos^3(x)dx$.

We cannot apply a substitution in this case because the derivative of the cosine function does not appear. However, we know that the fundamental relation of trigonometry $\cos^2(x)+\sin^2(x) = 1$ allows us to rewrite the square of a cosine in terms of sine. We can therefore proceed as follows.

$$\begin{aligned} \int \cos^3(x)dx &= \int \cos^2(x) \cdot \cos(x)dx \\ &= \int (1 - \sin^2(x)) \cos(x)dx \\ &= \int \cos(x)dx - \int \sin^2(x) \cos(x)dx. \end{aligned}$$

We already know how to integrate $\cos(x)$, so the first term is easily solved. For the second term, observe that $\cos(x)$ is the derivative of $\sin(x)$, so that we can perform the substitution $u = \sin(x)$, and then $\frac{du}{dx} = \frac{d}{dx}(\sin(x)) = \cos(x)$. We therefore have (by substitution):

$$\begin{aligned}\int \sin^2(x)\cos(x)dx &= \int u^2 du \\ &= u^3/3 + c \\ &= \frac{1}{3}\sin^3(x) + c,\end{aligned}$$

where in the last step we have substituted $u = \sin(x)$ back again. We therefore have

$$\int \cos^3(x)dx = \sin(x) - \frac{1}{3}\sin^3(x) + c.$$

Example 8.2.2. We now consider a higher power of a trigonometric function. We compute $\int \sin^4(x)dx$.

To solve this integral, we use a combination of integration by parts and trigonometric integrals.

We first observe that $\int \sin^4(x)dx = \int \sin^3(x)\sin(x)dx$. Then, we use the fact that $\sin(x)$ is the derivative of $-\cos(x)$ to integrate by parts and get

$$\begin{aligned}\int \sin^4(x)dx &= \int \sin^3(x)\sin(x)dx \\ &= -\sin^3(x)\cos(x) + 3\int \sin^2(x)\cos^2(x)dx.\end{aligned}$$

At this point it seems that we did not go really that far, since the last integral seems even more complicated than the initial one. However, rewriting $\cos^2(x) = 1 - \sin^2(x)$ we can use the same trick that we used for the integral of $e^x \sin(x)$.

In fact, with $\cos^2(x) = 1 - \sin^2(x)$ we have

$$\begin{aligned}\int \sin^4(x)dx &= -\sin^3(x)\cos(x) + 3\int \sin^2(x)\cos^2(x)dx \\ &= -\sin^3(x)\cos(x) + 3\int \sin^2(x)(1 - \sin^2(x))dx \\ &= -\sin^3(x)\cos(x) + 3\int \sin^2(x)dx - 3\int \sin^4(x))dx,\end{aligned}$$

from which we get

$$4\int \sin^4(x)dx = -\sin^3(x)\cos(x) + 3\int \sin^2(x)dx.$$

Therefore, our integral has reduced now to

$$\int \sin^4(x)dx = -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\int \sin^2(x)dx.$$

In other words, our problem is reduced to solving $\int \sin^2(x)dx$. Another integration by parts shows that $\int \sin^2(x)dx = \frac{1}{2}(x - \sin(x)\cos(x)) + c$.

We have therefore obtained

$$\int \sin^4(x)dx = -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{8}(x - \sin(x)\cos(x)) + c.$$

We now consider the general approach to solve integrals where products of sine and cosine functions appear. This has three subcases.

Method 8.2.3. *We want to solve integrals of type* $\int \sin^m(x)\cos^n(x)dx$.

(i) If n *is an odd number, we write* $\cos^n(x) = \cos^{n-1}(x)\cos(x)$. *Now,* $n-1$ *is even, so it can be written as* $n-1 = 2k$ *for some number* k. *We therefore have for our integral* $\int \sin^m(x)\cos^n(x)dx = \int \sin^m(x)\cos^{2k}(x)\cos(x)dx = \int \sin^m(x)(\cos^2(x))^k\cos(x)dx$. *Since* $\cos^2(x) = 1-\sin^2(x)$, *we can now solve this integral with a substitution of type* $u = \sin(x)$.

(ii) When the m *is odd, we proceed in the same case as above, but writing* $\sin^m(x)$ *as product of type* $(\sin^2(x))^k\sin(x)$. *This integral can then be solved using a substitution of type* $u = \cos(x)$.

(iii) If both n *and* m *are even, we can then use the half-angle formulas to rewrite even powers of sines and cosines as* $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$ *and* $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$.

In the next example, we show that even when the function that is being integrated is not trigonometric we can use a trigonometric substitution. This is a very useful trick, and it basically consists in using the integration by substitution in a reversed way with respect to how we have used so far.

Example 8.2.4. We want to compute $\int \frac{\sqrt{1-x^2}}{x^2}dx$.

We see that if $x = \sin(\theta)$, we would have $\sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)}$. This would be quite convenient, because we would be able to remove the square root. Setting $x = \sin(\theta)$ we also have $\frac{dx}{d\theta} = \frac{d}{d\theta}\sin(\theta) = \cos(\theta)$. So, using this substitution we will need to replace dx by $dx = \cos(\theta)d\theta$.

We now have

$$\begin{aligned}\int \frac{\sqrt{1-x^2}}{x^2}dx &= \int \frac{\sqrt{1-\sin^2(\theta)}}{\sin^2(\theta)}\cos(\theta)d\theta \\ &= \int \frac{\sqrt{\cos^2(\theta)}}{\sin^2(\theta)}\cos(\theta)d\theta \\ &= \int \frac{\cos^2(\theta)}{\sin^2(\theta)}d\theta.\end{aligned}$$

Now, observe that $\frac{\cos^2(\theta)}{\sin^2(\theta)} = \csc^2(\theta) - 1$, since we have $\csc^2(\theta) - 1 = \frac{1}{\sin^2(\theta)} - 1 = \frac{1-\sin^2(\theta)}{\sin^2(\theta)} = \frac{\cos^2(\theta)+\sin^2(\theta)-\sin^2(\theta)}{\sin^2(\theta)} = \frac{\cos^2(\theta)}{\sin^2(\theta)}$. So, we have

$$\begin{aligned}\int \frac{\sqrt{1-x^2}}{x^2}dx &= \int \frac{\cos^2(\theta)}{\sin^2(\theta)}d\theta \\ &= \int (\csc^2(\theta)-1)d\theta \\ &= -\cot(\theta) - \theta + c \\ &= -\cot(\arcsin(x)) - \arcsin(x) + c.\end{aligned}$$

This completes our computation. In fact, one can also show using the trigonometric formulas that $\cot(\theta) = \frac{\sqrt{1-x^2}}{x^2}$, which gives a simpler form to the previous solution.

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ $(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = \sec\theta$, $(0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2})$	$\sec^2\theta - 1 = \tan^2\theta$

Table 8.1: Trigonometric substitutions.

There a few useful rules for substitutions as in the example above. We summarize them in Table 8.1.

In the table, we report the type of expressions to solve by trigonometric substitution, the substitution, where we also give the domain of definition of the angle θ, and the type of identity that is used after substitution in order to simplify the integral. One can see that Example 8.2.4 uses the first type of substitution.

Example 8.2.5. We want to compute $\int \frac{1}{x^2\sqrt{x^2+4}}dx$.

From the table we see that this is an integral whose trigonometric substitution requires the use of tangent.

Therefore, we set $x = 2\tan\theta$. From this substitution we also find that $\frac{dx}{d\theta} = \frac{d}{d\theta}(2\tan\theta) = 2\sec^2\theta$. We therefore have $dx = 2\sec^2\theta d\theta$. Using the identity given in the table, we also see that $\sqrt{4 + 4\tan\theta} = \sqrt{4\sec^2\theta} = 2\sec\theta$. Our integral becomes

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2+4}}dx &= \int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}2\sec^2\theta d\theta \\ &= \int \frac{\sec^2\theta}{4\tan^2\theta\sec\theta}d\theta \\ &= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}d\theta.\end{aligned}$$

Since $\sec\theta = \frac{1}{\cos\theta}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$, we have $\frac{\sec\theta}{\tan^2\theta} = \frac{\cos\theta}{\sin^2\theta}$. We therefore need to solve the integral $\int \frac{\sec\theta}{\tan^2\theta}d\theta$. To achieve this we can perform the substitution $u = \sin\theta$, since $\cos\theta$ is the derivative of $\sin\theta$. We get $\int \frac{\sec\theta}{\tan^2\theta}d\theta = -\frac{1}{u} + c$.

Therefore, for the initial integral, we have

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2+4}}dx &= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}d\theta \\ &= -\frac{1}{4}\frac{1}{u} + c \\ &= -\frac{1}{4}\frac{1}{\sin\theta} + c.\end{aligned}$$

This solves the integration problem. As in the previous example, we can obtain the solution in terms of the original variable x by using the trigonometric identities. We have that $x = 2\tan\theta$, which means that we can construct a right triangle where θ is one of the angles, x is the side opposite to θ, 2 is the side adjacent to θ, and $\sqrt{x^2+4}$ is the hypotenuse. In this case, $\sqrt{x^2+4}\cdot\sin\theta = x$, from which it follows that $\csc x\theta = \frac{1}{\sin\theta} = \frac{\sqrt{x^2+4}}{x}$. So, the answer can be written in terms of x as $\int \frac{1}{x^2\sqrt{x^2+4}}dx = -\frac{\sqrt{x^2+4}}{4x} + c$.

8.3 Partial Fractions

The method of partial fractions allows us to integrate rational functions, i.e. functions of type $f(x) = \frac{P(x)}{Q(x)}$ where numerator and denominator, $P(x)$ and $Q(x)$ respectively, are polynomials.

Polynomials are described by a formula of type $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n \neq 0$. The number n here is called degree of P, and indicated by the symbol $\deg(P)$. Our problem is to obtain $\int \frac{P(x)}{Q(x)}dx$, where both P and Q are polynomials.

Suppose first that the numerator has degree larger than or equal to the degree of the denominator, i.e. $\deg(P) \geq \deg(Q)$. It is a known fact that we can write this quotient as $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$, where both $S(x)$ and $R(x)$ are polynomials, and $R(x)$ has degree lower than the degree of $Q(x)$ (if $\deg Q > 0$ also $\deg S < \deg Q$). The polynomial $R(x)$ is called the remainder of the long division.

Example 8.3.1. Compute: $\int \frac{x^3+x}{x-1}dx$.

Observe that the numerator has a power larger than the denominator. So, we can use the long division to write $\frac{x^3+x}{x-1} = S(x) + \frac{R(x)}{x-1}$. In other words, we need to find S and R. Using long division, here we find that $S(x) = x^2 + x + 2$, and $R(x) = 2$. Therefore, we have

$$\begin{aligned} \int \frac{x^3+x}{x-1}dx &= \int [x^2 + x + 2 + \frac{2}{x-1}]dx \\ &= \frac{x^3}{2} + \frac{x^2}{2} + 2x + 2\ln|x-1| + c. \end{aligned}$$

Now on, we will consider fractions where the degree of the numerator is smaller than the degree of the denominator. In fact, if this is not the case, by long division we can reduce our integration to an integral of type $\in S(x)dx + \in \frac{R(x)}{Q(x)}dx$, where the first integral is eqsily solvable, and the secon integral is a fraction where the degree of the numerator is smaller than the degree of the denominator.

The previous example is simple because the denominator does not contain higher powers of x. In general, this is not the case. However, we might want to factor the denominator $Q(x)$ as much as possible.

In general, a polynomial such as $Q(x)$ can always be written as a product of linear terms and irreducible quadratic terms. Linear terms are polynomials of type $ax+b$, while irreducible quadratic terms are of type $ax^2 + bx + c$, where $b^2 - 4ac < 0$. Observe that if $b^2 - 4ac \geq 0$, we can factor the quadratic term into a product of two linear terms. Depending on how $Q(x)$ factors in a product of linear and quadratic terms, there are several cases in which we can split our approach to integration.

8.3.1 $Q(x)$ is a product of linear factors with no repetitions

If $Q(x)$ is just a product of linear factors, with no linear factor repeated, we have $Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$ for some numbers $a_1, a_2, \ldots a_k$, and $b_1, b_2, \ldots b_k$. In such a case, we can decompose the fraction as $\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$, for constants A_i that we need to find.

Example 8.3.2. We want to compute $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$.

The denominator $Q(x) = 2x^3 + 3x^2 - 2x$ factors into linear terms, since $2x^3 + 3x^2 - 2x = x(2x-1)(x+2)$. We want to write $\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$. Taking the common denominator on the right hand side, we find

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)}{x(2x-1)(x+2)}.$$

This equality holds if and only if the numerators are the same, so that we have found that

$$x^2+2x-1 = A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1) = (2A+B+2C)x^2+(3A+2B-C)x-2A.$$

This produces a system of type

$$\begin{cases} 2A+B+2C = 1 \\ 3A+2B-C = 2 \\ -2A = -1 \end{cases}$$

We therefore find $A+1/2$, $B=1/5$ and $C=-1/10$. We can therefore write the integral as

$$\begin{aligned} \int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx &= \int [\frac{1}{2x} + \frac{1}{5(2x-1)} - \frac{1}{10(x+2)}]dx \\ &= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + c. \end{aligned}$$

8.3.2 $Q(x)$ is a product of linear factors with repetitions

In this case, $Q(x)$ is again a product of linear factors, but some of these factors are repeated. We can therefore write $Q(x)$ as:

$$Q(x) = (a_1x+b_1)^{r_1}(a_2x+b_2)^{r_2}\cdots(a_kx+b_k)^{r_k},$$

where at least one of the numbers $r_1, r_2, \ldots, r_k$ is larger than 1. Suppose that we have r_1 strictly larger than 1, for the sake of simplicity. Then, in the fraction decomposition given above, we have to replace the term $\frac{A_1}{a_1x+b_1}$ by the fractions $\frac{A_{11}}{a_1x+b_1} + \frac{A_{12}}{(a_1x+b_1)^2} + \cdots + \frac{A_{1r_1}}{(a_1x+b_1)^{r_1}}$. We therefore have

$$\begin{aligned} \frac{P(x)}{Q(x)} &= \frac{A_{11}}{a_1x+b_1} + \frac{A_{12}}{(a_1x+b_1)^2} + \cdots + \frac{A_{1r_1}}{(a_1x+b_1)^{r_1}} + \cdots \\ &\cdots + \frac{A_{k1}}{a_kx+b_1} + \frac{A_{k2}}{(a_kx+b_k)^2} + \cdots + \frac{A_{kr_k}}{(a_kx+b_k)^{r_k}}. \end{aligned}$$

In other words, for all those factors where the linear factor appears only once, we will add a factor as before of type $\frac{A_t}{(a_tx+b_t)}$, while whenever we have a linear term that appears multiple times, we add several fractions of type $\frac{A_{11}}{a_1x+b_1} + \frac{A_{12}}{(a_1x+b_1)^2} + \cdots + \frac{A_{1r_1}}{(a_1x+b_1)^{r_1}}$. The constants A_{ij} need to be obtained through a direct computation as in Example 8.3.2.

We show this method with an example.

Example 8.3.3. We want to compute the integral: $\int \frac{x+2}{x^3-x^2-x+1}dx$.

First of all, we need to factor the denominator, and understand its factorization into products of linear and quadratic factors. We see immediately that $x=1$ is a root of $Q(x)=x^3-x^2-x+1$. So, the term $x-1$ divides $Q(x)$. From long division we see that $Q(x)=(x-1)(x^2-1)$. Since

$x^2 - 1 = (x-1)(x+1)$, we have found that $Q(x) = (x-1)^2(x+1)$, which shows that Q factors into linear terms, one of which is repeated twice. It follows that we seek to write the fraction as

$$\frac{x+2}{x^3-x^2-x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}.$$

Doing the common fraction we have

$$\begin{aligned}\frac{x+2}{x^3-x^2-x+1} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \\ &= \frac{A(x-1)(x+1) + B(x+1) + C(x-1)^2}{(x-1)^2(x+1)} \\ &= \frac{Ax^2 - A + Bx + B + Cx^2 + C - 2Cx}{(x-1)^2(x+1)} \\ &= \frac{(A+C)x^2 + (B-2C)x + (-A+B+C)}{(x-1)^2(x+1)}.\end{aligned}$$

This gives us the system

$$\begin{cases} A + C = 0 \\ B - 2C = 1 \\ -A + B + C = 2 \end{cases}$$

which has the unique solution $-A = C = \frac{1}{4}$, $B = \frac{3}{2}$. Therefore, our fraction is written as

$$\frac{x+2}{x^3-x^2-x+1} = -\frac{1}{4(x-1)} + \frac{3}{2}\frac{1}{(x-1)^2} + \frac{1}{4(x+1)}.$$

The integral is therefore rewritten as

$$\begin{aligned}\int \frac{x+2}{x^3-x^2-x+1}dx &= -\int \frac{1}{4(x-1)}dx + \frac{3}{2}\int \frac{1}{(x-1)^2}dx + \int \frac{1}{4(x+1)}dx \\ &= -\frac{1}{4}\ln|x-1| - \frac{3}{2}\frac{1}{x-1} + \frac{1}{4}\ln|x+1| + c.\end{aligned}$$

8.3.3 $Q(x)$ contains irreducible quadratic factors without repetitions

Let us now consider the case when the factorization of $Q(x)$ contains quadratic factors of type $ax^2 + bx + c$ where $b^2 - 4ac < 0$, and therefore we cannot decompose it into a product of linear factors. In this case, in addition to the sum of fractions considered before, we also need to consider fractions of type $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants that need to be found. Of course, we need to understand how to integrate a fraction of type $\frac{Ax+B}{ax^2+bx+c}$, since once we find A and B, our integral will contain a summand of this type, which needs to be integrated. Observe that $\frac{Ax+B}{ax^2+bx+c} = \frac{Ax}{ax^2+bx+c} + \frac{B}{ax^2+bx+c}$. The first summand is easy to integrate (why?). The second summand is obtained through a trigonometric integration. First one completes the square to have a denominator of type $\frac{1}{x^2+d^2}$, and then this integral is given by

$$\int \frac{1}{x^2+d^2} = \frac{1}{d}\tan^{-1}\left(\frac{x}{d}\right) + c. \tag{8.7}$$

We illustrate this with a computational example.

Example 8.3.4. We want to compute $\int \frac{2x^2-x+4}{x^3+4x}dx$.

First, we need to factor the denominator. We have $x^3+4x = x(x^2+4)$. The quadratic term x^2+4 does not simplify any further, since it does not have any real roots or, equivalently, the discriminat $\Delta = b^2-4ac < 0$. We therefore have a quadratic term, and a linear term. We already know how to deal with linear terms, but we need to apply the procedure described right above this example to the quadratic terms.

We seek to find constants A, B and C such that

$$\frac{2x^2-x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}.$$

Doing the common fraction we obtain

$$\frac{2x^2-x+4}{x^3+4x} = \frac{(A+B)x^2+Cx+4A}{x(x^2+4)}.$$

Equating the terms in the numerator, and solving for A, B and C we obtain $A=1$, $B=1$ and $C=-1$. Therefore our integral has become

$$\begin{aligned}\frac{2x^2-x+4}{x^3+4x}dx &= \int \frac{1}{x}dx + \int \frac{x-1}{x^2+4}dx\\ &= \int \frac{1}{x}dx + \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx\\ &= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}(\frac{x}{2}) + c,\end{aligned}$$

where we have performed the substitution $u = x^2+4$ for the second integral, and removed the absolute value in the logarithm because x^2+4 is always strictly positive.

8.3.4 $Q(x)$ contains irreducible quadratic factors with repetitions

In this case we have $Q(x)$ factoring into a product where some term of type $(ax^2+bx+c)^r$ appears, where $r > 1$ and $b^2-4ac < 0$. In this case, similar to how we proceeded for the case of repeated linear factors we have to add fractions of type $\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$. Then, we have to determine the coefficients as before. These terms can all be integrated by possibly completing the square at the denominator, and using a substitution so, once we reach such a form for our fraction, we can integrate.

Example 8.3.5. We want to compute $\int \frac{x^2+2}{x(x^2+1)^2}dx$.

Our denominator is already factored, and we know that the quadratic term cannot be simplified any further. We seek to find constants such that the equality

$$\frac{x^2+2}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

As usual, by doing the common fraction and then equating the terms we obtain a system, which gives us $A = 2 = -B$, $C = E = 0$, and $D = -1$. We therefore have for the integral

$$\begin{aligned}\int \frac{x^2+2}{x(x^2+1)^2}dx &= \int \frac{2}{x}dx - \int \frac{x}{x^2+1}dx - \int \frac{x}{(x^2+1)^2}dx\\ &= 2\ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2}(x^2+1)^{-1} + c.\end{aligned}$$

8.4 Numerical Integration

In most of practical applications, e.g. in physics, computer science and engineering, it is not convenient, or even possible, to perform exact integration. This problem requires a numerical approach to integration, where we are able to approximate integrals without exactly computing them.

Recall, that the definition of (definite) integral consist of a limit of approximations (the Riemann sums), which become increasingly accurate when the number of subintervals grows. Therefore, taking a Riemann sum with n subintervals is a way of approximating an integral. When we introduced integrals we discussed three simple choices for the computation of Riemann sums, namely left, right and mid rules. We use them to obtain two important approaches to numerical integration: Midpoint rule, and trapezoidal rule.

8.4.1 Midpoint Rule

For the midpoint rule, for each subinterval of size Δx we pick the point in the middle, and use this to create our Riemann sum. Suppose that we want to approximate the integral $\int_a^b f(x)dx$. Then, we divide $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, where the points x_i are chosen inside $[a, b]$ and are assumed to be equally spaced. Then, we set $\bar{x}_i = \frac{x_i - x_{i-1}}{2}$ for all $i = 1, 2, \ldots, n$. The corresponding Riemann sum is then given by

$$\int_a^b f(x)dx \approx \Delta x[f(\bar{x}_1) + \cdots + f(\bar{x}_n)] := M_n, \tag{8.8}$$

where $\Delta x = \frac{b-a}{n}$ is the size of the subintervals.

When performing an approximated computation, one is interested in understandig that we make in our approximation, or better to say an estimate for the error. This means that we want to understand what is the largest error that we can make when performing numerical integration through the midpoint rule. The error is defined as

$$E_M = |\int_a^b f(x)dx - M_n|.$$

One can prove (but we will not do so here) that if $f(x)$ is twice differentiable and $|f''(x)| \leq K$ for all x in $[a, b]$ and for some number K, then we have the following error estimate for the midpoint rule:

$$E_M \leq \frac{K(b-a)^3}{24n^2}.$$

This shows that as we increase the number n of points used to subdivide the interval $[a, b]$, the precision of our integration increases, since the error E_M becomes smaller and smaller.

8.4.2 Trapezoidal Rule

The trapezoidal rule is defined as the average of the left and right Riemann sums. Recall that if n is the number of subintervals we divide $[a.b]$ into, then the left and right Riemann sums for the integral $\int_a^b f(x)dx$ are given by

$$\int_a^b f(x)dx \approx \Delta x[f(x_0) + f(x_1) + \cdots + f(x_{n-1})] =: L_n \tag{8.9}$$

$$\int_a^b f(x)dx \approx \Delta x[f(x_1 1) + f(x_2) + \cdots + f(x_n)] =: R_n. \tag{8.10}$$

The trapezoidal rule uses both these estimates for $\int_a^b f(x)dx$, by taking their average. So, we have

$$\int_a^b f(x)dx \approx \frac{L_n + R_n}{2} \tag{8.11}$$

$$= \frac{1}{2}\{\Delta x[f(x_0) + f(x_1) + \cdots + f(x_{n-1})] + \Delta x[f(x_1 1) + f(x_2) + \cdots + f(x_n)]\} \tag{8.12}$$

$$= \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]. \tag{8.13}$$

Again, we are interested in the error

$$E_T = |\int_a^b f(x)dx - \frac{L_n + R_n}{2}|.$$

Similarly to the case of the midpoint rule, we have a bound on how big the error can be in approximating an integral with the trapezoidal rule. For $f(x)$ such that $|f''(x)| \leq K$ for all x in $[a, b]$, and some K, we get

$$E_T \leq \frac{K(b-a)^3}{12n^2}.$$

Once again, as n grows, our error becomes smaller and smaller.

8.4.3 Cavalieri-Simpson's Rule

This rule was discovered by Bonaventura Cavalieri in the 1600's, and rediscovered by Thomas Simpson in the 1700's. The method consists of approximating a given function between two points using a parabola, and performing integration using a rule for the parabola. If the points used are close enough, the result can be quite accurate.

We divide the interval $[a, b]$ into n subintervals of the same length, as before, with the extra constraint that n be even. We also set $\Delta x = \frac{b-a}{n}$. We have the $n+1$ points $a = x_0, x_1, \ldots, x_{n-1}, x_n = b$ in the interval $[a, b]$ corresponding to the chosen partition. Between each pair of points $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$ we approximate the function f by means of a parabola in order to perform integration.

To do so, we consider triple of points $(x_i, f(x_i)), (x_{i+1}, f(x_{i+1}))$ and $(x_{i+2}, f(x_{i+2}))$, and find the equation of a parabola passing through them. We derive the equations for the case where the points are centered around $x = 0$, since upon translating this result, we can then generalize it to arbitrary triples. So, we assume we have $(-h, f(-h))), (0, f(0))$ and $(h, f(h))$. A parabola has the form $y = ax^2 + bx + c$, where a, b, c are numbers chosen so that the parabola passes through the three points.

We can therefore compute the integral of the parabola from $-h$ to h and obtain

$$\begin{aligned}\int_{-h}^{h}(ax^2+bx+c)dx &= 2\int_{0}^{h}(ax^2+c)dx\\ &= 2[a\frac{x^3}{3}+cx]_0^h\\ &= 2(a\frac{x^3}{3}+cx)\\ &= \frac{h}{3}(2ah^2+6c),\end{aligned}$$

where in the second equality we have used the fact that $\int_{-h}^{h} bxdx = 0$ (the function is odd!), and have therefore removed bx from the integration, along the fact that $\int_{-h}^{h}(ax^2+c)dx = 2\int_0^h(ax^2+c)dx$, since the function ax^2+c is even.

using the fact that the parabola passes through the points $(-h, f(-h)))$, $(0, f(0))$ and $(h, f(h))$, we find the system

$$\begin{cases} f(-h) = a(-h)^2 - bh + c\\ f(0) = c\\ f(h) = ah^2 + bh + c\ . \end{cases}$$

By summing first equation, 4 times second equation, and third equation we obtain the equality $f(-h)+4f(0)+f(h) = 2ah^2+6c$, which is precisely the term in the parenthesis appearing in the evaluation of the integral above. We can therefore write the integral above as

$$\int_{-h}^{h}(ax^2+bx+c)dx = \frac{h}{3}(f(-h)+4f(0)+f(h)).$$

When dealing with arbitrary points $(x_i, f(x_i)), (x_{i+1}, f(x_{i+1}))$ and $(x_{i+2}, f(x_{i+2}))$, the procedure is precisely the same, just replacing x_i to $-h$, x_{i+1} to 0, and x_{i+2} to h. So we get in general an area of

$$\int_{x_i}^{x_{i+2}}(ax^2+bx+c)dx = \frac{h}{3}(f(x_i)+4f(x_{i+1})+f(x_{i+2})).$$

When computing the full integral $\int_a^b f(x)dx$, we can therefore approximate it by summing the integrals between all pairs of even points x_0 to x_2, x_2 to x_4 and so on, using the integration formula obtained above. In fact, observe that we will have for each subinterval a triple of points exactly as above. We therefore obtain

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)],$$

where we have used Δx instead of h, since this is the length of the subintervals.

Example 8.4.1. Let us consider the integral $\int_1^2 \frac{1}{x}dx$. We take $n = 10$, which gives us $\Delta x = 0.1$. Here we have $x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$ and so on up to $x_9 = 1.9$ and $x_{10} = 2$. Using the Cavalieri-Simpson's rule we have the approximation

$$\int_1^2 \frac{1}{x}dx \approx \frac{0.1}{3}[1+\frac{4}{1.1}+\frac{2}{1.2}+\frac{4}{1.3}+\frac{2}{1.4}+\frac{4}{1.5}+\frac{2}{1.6}+\frac{4}{1.7}+\frac{2}{1.8}+\frac{4}{1.9}+\frac{1}{2}] \approx 0.693150.$$

Observe that the precise result, after integrating analytically, is $\int_1^2 \frac{1}{x}dx = \ln(2) - \ln(1) = \ln(2) \approx 0.69314718056$. So, the Cavalieri-Simpson's rule gives a very good approximation of the integral.

Similarly to the case of midpoint and trapezoidal rules, we are interested in finding error bounds for the numerical integration. One can prove that the error

$$E_{CS} \leq \frac{K(b-a)^5}{180n^4}$$

holds, where K is a number such that $|f^{(4)}(x)| \leq K$ for all x in $[a,b]$.

8.5 Improper Integrals

Our definition of definite integral was based on the fact that we started with a function that is defined on a closed and finite interval of type $[a,b]$. One might ask whether there is a way of defining, or at least making sense of, integrals on intervals of type $[a,\infty)$, or $[a,b)$ etc.

In this section we will see that we can indeed do that. Such integrals are called improper integrals.

8.5.1 Improper integrals on infinite intervals

We start by considering the case of integrals over infinite intervals, and we consider an example first. This will make the definition of improper integral very clear.

Example 8.5.1. Let $f(x) = \frac{1}{x^2}$. We want to compute the area beneath $f(x)$ between $x = 1$ and ∞. In other words, we want to compute a reaasonable notion of $\int_1^\infty f(x)dx$. One natural way to do this would be to compute $\int_1^t f(x)dx$ for an arbitrary $t > 1$, and then take the limit $t \longrightarrow \infty$ of the quantity that we obtain. In fact, computing $\int_1^t f(x)dx$ would give us a notion of area under $f(x)$ "up to t", and then by letting t go to infinity, we consider areas that are larger and larger without putting a bound on it. Of course, one might intuitively expect that this would give us a quantity that is infinite in magnitude. However, this is not the case, as we will see shortly.

We compute

$$\begin{aligned} A(t) &= \int_1^t \frac{1}{x^2}dx \\ &= = -\frac{1}{x}]_1^t \\ &= 1 - \frac{1}{t}. \end{aligned}$$

So, the area $A(t)$ is always less than 1! Moreover, when we take the limit $t \longrightarrow \infty$, we get $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} 1 - \frac{1}{t} = 1$, which is not infinite as one might have expected. This prompts us to think of the equality (which is a definition) $\int_1^\infty \frac{1}{x^2}dx = \lim t \to \infty \int_1^t \frac{1}{x^2}dx$.

In the previous example, there is an important fact that has appened behind the scenes. We were able to compute $\int_1^\infty \frac{1}{x^2}dx$ for all $t > 1$. This is important, because we need to be able to define $A(t)$. We pose now the following definition, based on the previous observations.

Definition 8.5.2. Suppose that $\int_a^t f(x)dx$ exists for all $t \geq a$. Then, we define

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx,$$

in the assumption that the limit exists (either finite or infinite). We also similarly define $\int_{-\infty}^b f(x)dx$. When the integral is finite, we say that the improper integral is *convergent*, while when the limit exists but is infinite, we say that the improper integral is *divergent*, and we write $\int_a^\infty f(x)dx = \pm\infty$ (depending on the sign). If the limit does not exist at all, then one says that the integral is *oscillatory divergent*, and we do not assign any value to $\int_a^\infty f(x)dx$, either finite or infinite.

Moreover, if $\int_{-\infty}^a f(x)dx$ and $\int_a^\infty f(x)dx$ are finite, i.e. the integrals are convergent, we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx.$$

Example 8.5.3. Let us now compute the integral $\int_1^\infty \frac{1}{x}dx$, if it exists. We compute

$$\begin{aligned}\int_1^\infty \frac{1}{x}dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x}dx \\ &= \lim_{t\to\infty} \ln|x|]_1^t \\ &= \lim_{t\to\infty} (\ln t - \ln 1) \\ &= \lim_{t\to\infty} \ln t \\ &= \infty.\end{aligned}$$

So, this integral is divergent.

More generally, we can compute $\int_1^\infty \frac{1}{x^p}dx$ for some $p \neq 1$. In this case we have

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x^p}dx \\ &= \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}]_1^t \\ &= \lim_{t\to\infty} \frac{1}{p-1}[\frac{1}{t^{p-1}} - 1].\end{aligned}$$

Now, observe that if $p > 1$, we have that $\frac{1}{t^{p-1}} \longrightarrow 0$ for $t \longrightarrow \infty$. So, in this case the integral is convergent and we have $\int_1^\infty \frac{1}{x^p}dx = \frac{1}{1-p}$. However, when $p < 1$ we have that $\frac{1}{t^{p-1}} \longrightarrow \infty$, and therefore the integral is divergent to ∞.

Considering also the case $p = 1$ computed before, we have that $\int_1^\infty \frac{1}{x^p}dx = \frac{1}{1-p}$ when $p > 1$, while $\int_1^\infty \frac{1}{x^p}dx$ is divergent for $p \leq 1$.

8.5.2 Finite non-closed improper integrals

In this case the interval of integration is finite, but since the function is defined over an open or half-open interval, our function might diverge to $\pm\infty$. One can think of the situation where we want to integrate $\frac{1}{x}$ between 0 and 1. The idea to treat this situation is similar to the previous case, and we pose the following definition.

Definition 8.5.4. Suppose that f is continuous on $[a,b)$, and possibly not defined or discontinuous at $x=b$ Then, we define

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx,$$

if the limit exists (finite or infinte). A similar definition holds when f is continuous on $(a,b]$, and it is not defined or it is discontinuous aat $x=a$. We can pose similar notation for convergent, divergent, and oscillatory divergent improper integrals.

In addition, when f has a point c of discontinuity or where it is not defined, with c in $[a,b]$, we can define the integral as

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Example 8.5.5. We compute the integral $\int_2^5 \frac{1}{\sqrt{x-2}}dx$. Since the function $f(x)=\frac{1}{\sqrt{x-2}}$ is not defined at $x=2$, we need to use Definition 8.5.4.

We compute

$$\begin{aligned}\int_2^5 \frac{1}{\sqrt{x-2}}dx &= \lim_{t\to 2^+}\frac{1}{\sqrt{x-2}}dx\\ &= \lim_{t\to 2^+} 2\sqrt{x-2}]_t^5\\ &= \lim_{t\to 2^+} 2(\sqrt{3}-\sqrt{t-2})\\ &= 2\sqrt{3}.\end{aligned}$$

So, the integral is convergent.

Chapter 9

Arc Length, Areas, and Applications

In this chapter we consider further applications of integration to physics and engineering. Before doing so, we introduce two important concepts, which use integration methods, and then apply them to some real-world cases.

9.0.1 Arc Length

We know how to find the lenght of curves that are polygons, simply by adding all the lenghts of the segments that form the polyogon. However, this simply procedure does not apply when the curve is more general, and is for instance defined by some function through an equation of type $y = f(x)$. The idea to introduce the notion of length in such more general cases is similar to how we defined the notion of integral as area under a curve. We reduce the problem to something we know how to compute, i.e. polygonal curves, and then use them to define the length of a general curve through a procedure of limit.

Suppose $y = f(x)$ is a curve C defined through a function f which is continuous, and let x be in $[a, b]$. We define a polygonal approximation of C by choosing points $a = x_0, x_1, \dots, x_{n-1}, x_n = b$ on the interval, and taking the corresponding coordinates $f(x_i)$ on the curve C. So, we get points $P_i = (x_i, f(x_i))$ along the curve. Now, we can join the points P_i so to obtain a polygonal curve which approximates the original curve. The approximated length of C, is given by the length of the polygonal curve as

$$L_n = \sum_{i=1}^{n} |P_i - P_{i-1}|, \tag{9.1}$$

where $|P_i - P_{i-1}|$ indicates the length of the segment joining P_{i-1} and P_i. Then, we define the length of C by taking the limit as n goes to infinity, i.e. we sample more and more points along the curve so that the polygonal approximation becomes more precise:

$$L = \lim_{n \to \infty} L_n. \tag{9.2}$$

When f has a continuous, we can derive a more useful formula for Equation (9.2), by rewriting the terms in Equation (9.1) in a more convenient way. In fact, if $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1} = f(x_i) - f(x_{i-1})$, the distance $|P_i - P_{i-1}|$ is given by

$$|P_i - P_{i-1}| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}. \tag{9.3}$$

Applying the Mean Value Theorem we find that

$$f(x_i) - f(x_{i-1}) = f'(x*_i)(x_i - x_{i-1}), \tag{9.4}$$

for some point x_i^* in $[x_{i-1}, x_i]$, which gives us

$$\Delta y_i = f'(x_i^*)\Delta x_i. \tag{9.5}$$

Since the points x_i can be chosen to be equally spaced, we can also drop the subscript i for Δx_i. We therefore have for Equation (9.3)

$$\begin{aligned} |P_i - P_{i-1}| &= \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2} && (9.6)\\ &= \sqrt{1 + [f'(x_i^*)]}\Delta x, && (9.7)\end{aligned}$$

and using this into Equation 9.2, we find

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=1}^{n} |P_i - P_{i-1}| && (9.8)\\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]}\Delta x && (9.9)\\ &= \int_a^b \sqrt{1 + [f'(x)]^2} dx. && (9.10)\end{aligned}$$

Equation (9.10) is called the *Arc Length formula*.

Example 9.0.1. We want to find the lenght of the arc from $(1, 1)$ to $(4, 8)$ on the curve defined through the equation $y^2 = x^3$.

First, since both x and y are positive on the arc between $(1, 1)$ to $(4, 8)$ (the curve lies on the upper half of the plane), we can simply write $y = x^{\frac{3}{2}}$, by taking square roots of both sides without introduce any minus sign. Then, we can find the derivative of y with respect to x, since it is needed in Equation (9.10). We have $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$. We therefore have

$$\begin{aligned} L &= \int_a^b \sqrt{1 + [f'(x)]^2} dx\\ &= \int_1^4 \sqrt{1 + \frac{9}{4}x} dx.\end{aligned}$$

We now perform the substitution $u = 1 + \frac{9}{4}x$ and solve the integral, finding $L = \frac{1}{27}(80\sqrt{10} - 13\sqrt{13})$.

If the equation of the curve is such that we are able to write x as a function of y, then we can proceed exactly in the same way as before, but now exchanging the roles of x and y.

Example 9.0.2. Consider the equation of the parabola $x = y^2$. Suppose that we want to compute the lenght of the arc on the curve between $(0, 0)$ and $(1, 1)$. Since the curve is already given as a function of x with respect to the variable y, we can apply Equation (9.10) but where the roles of x and y are exchanged. Observe that $\frac{dx}{dy} = 2y$.

We compute

$$\begin{aligned} L &= \int_a^b \sqrt{1+[f'(y)]^2}dy \\ &= \int_0^1 \sqrt{1+4y^2}dy. \end{aligned}$$

To solve this integral, we need to perform a trigonometric substitution. In fact, if we set $2y = \tan\theta$, i.e. $y = \frac{1}{2}\tan\theta$, we find $dy = \frac{1}{2}^2\theta d\theta$, and also $\sqrt{1+4y^2} = \sqrt{1+\tan^2\theta} = \sec\theta$. Therefore, we need to compute the integral

$$\begin{aligned} L &= \int_0^1 \sqrt{1+4y^2}dy \\ &= \frac{1}{2}\int_0^{\arctan 2} \sec^3\theta d\theta \\ &= \sec\theta\tan\theta]_0^{\arctan 2} - \int_0^{\arctan 2} \sec\theta\tan^2\theta d\theta \\ &= \sec\theta\tan\theta]_0^{\arctan 2} - \int_0^{\arctan 2} \sec\theta(\sec^2\theta - 1)d\theta \\ &= \sec\theta\tan\theta]_0^{\arctan 2} - \int_0^{\arctan 2} \sec^3\theta d\theta + \int_0^{\arctan 2} \sec\theta d\theta. \end{aligned}$$

From this we obtain

$$\begin{aligned} L &= \frac{1}{4}[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|]_0^{\arctan 2} \\ &= \frac{1}{4}[2\sec\arctan 2 + \ln|\sec\arctan 2 + 2|], \end{aligned}$$

which completes the computation.

9.1 Area of Surface of Revolution

The intuitive idea is that if we have a curve, and we let this rotate about an axis, we will obtain a surface that bounds a solid. Similarly to how we can compute the lateral area of a cylinder from knowledge of the length of a side of the cylinder, and the radius of rotation, we can expect that a similar computation is applicable to the case of more general curves. However, the issue is that our curve is in general not straight, and therefore the same procedure is not applicable in the exact way because the radius of rotation depends on the position on the curve.

We can tackle the problem by subdividing the curve into smaller pieces by taking points $P_0, P_1, \ldots, P_n$ on the curve, so that with some approximation, we have a cylindrical surface generated by a segment of length $|P_i - P_{i-1}|$. The points P_i can be assumed to correspond to the points x_i on the interval $[a, b]$ of definition of the function f that gives the curve. Rotating the segment from P_{i-1} to P_i, we obtain a band (i.e. a small "cylinder") with slant height $l_i = |P_i - P_{i-1}|$, and average radius given by $\frac{y_{i-1}+y_i}{2}$. The surface is going to be given by $\pi(y_{i-1}+y_i)|P_i - P_{i-1}|$. Applying the same procedure that we used for the computation of arc length, we can find a point x_i^* such that $|P_i - P_{i-1}| = \sqrt{1+[f'(x_i^*)]^2}\Delta x$. We can, in addition, make the approximation where

$f(x_i^*) \approx y_{i-1}$ and $f(x_i^*) \approx y_i$, due to the fact that the length is assumed to be very small. So, we have found the formula

$$\pi(y_{i-1}+y_i)|P_i - P_{i-1}| = 2\pi\frac{y_{i-1}+y_i}{2}|P_i - P_{i-1}| \approx 2\pi f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\Delta x.$$

When we want to compute the whole are, we have to perform a limit for this procedure, and sum all the components, so that we define the area as the integral

$$S = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}dx = 2\pi\int_a^b y\sqrt{1+[\frac{dy}{dx}]^2}dx \tag{9.11}$$

The formula can be expressed, symbolically, in terms of infinitesimal arc length as

$$S = 2\pi\int yds,$$

where $y = f(x)$ is the function whose rotation gives the surface, and $ds = \sqrt{1+[f'(x)]^2}dx$ is the length of an infinitesmial arc.

Example 9.1.1. Consider the curve given by $y = \sqrt{4-x^2}$ with x in $[-1,1]$. Then, we have $\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}}$. For the surface we therefore find

$$\begin{aligned} S &= 2\pi\int_{-1}^1 \sqrt{4-x^2}\sqrt{1+[\frac{-x}{\sqrt{4-x^2}}]^2}dx \\ &= 2\pi\int_{-1}^1 \sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}dx \\ &= 4\pi\int_{-1}^1 dx \\ &= 8\pi. \end{aligned}$$

For a rotation about the $\vec{y}$-axis, we can proceed in the same way, finding an equation similar to Equation (9.11), but with x and y swapped:

$$S = 2\pi\int_a^b f(y)\sqrt{1+[f'(y)]^2}dy = 2\pi\int_a^b x\sqrt{1+[\frac{dx}{dy}]^2}dy, \tag{9.12}$$

where here x is a function of y. We can write this equation in terms of the infinitesimal arc length as well as

$$S = 2\pi\int xds. \tag{9.13}$$

We can also have a rotation about the $\vec{x}$-axis, and a curve described by an equation of type $x = g(y)$, with y between a and b. In this case, the surface area is given by

$$S = 2\pi\int_a^b g(y)\sqrt{1+[g'(y)]^2}dy = 2\pi\int_a^b y\sqrt{1+[\frac{dx}{dy}]^2}dy. \tag{9.14}$$

Using the infinitesimal arc length notation, we have

$$S = 2\pi\int yds, \tag{9.15}$$

	Rotation about $\vec{x}$	Rotation about $\vec{y}$
$y = f(x)$	$2\pi \int_a^b y\sqrt{1+[\frac{dy}{dx}]^2}dx$	$2\pi \int_a^b x\sqrt{1+[\frac{dy}{dx}]^2}dx$
$x = g(y)$	$2\pi \int_a^b y\sqrt{1+[\frac{dx}{dy}]^2}dy$	$2\pi \int_a^b x\sqrt{1+[\frac{dx}{dy}]^2}dy$

Table 9.1: Surface area integrals.

where now $ds = \sqrt{1+[\frac{dx}{dy}]^2}dy$. Once again, the whole discussion can be done with the roles of x and y exchanged.

To summarize, we have two types of equations. Equation (9.13) and Equation (9.15) where in both cases we can use either $ds = \sqrt{1+[\frac{dy}{dx}]^2}dx$ or $ds = \sqrt{1+[\frac{dx}{dy}]^2}dy$. We have Table 9.1 to remember what to use.

Example 9.1.2. Consider now the surface generated by rotating the function $y = x^2$ between $x = 1$ and $x = 2$ about the $\vec{y}$-axis. We want to compute the area of the surface of rotation.

We see that we are in the situation where we have a rotation about $\vec{y}$, and the curve is described by a function $y = f(x)$. So, we need to use the top-right entry of Table 9.1. So, we need to compute the integral $2\pi \int_a^b x\sqrt{1+[\frac{dy}{dx}]^2}dx$, where $y = x^2$. We have $\frac{dy}{dx} = 2x$, and $a = 1$, $b = 2$.

We have

$$S = 2\pi \int_1^2 x\sqrt{1+[2x]^2}dx \qquad (9.16)$$

$$= 2\pi \int_1^2 x\sqrt{1+4x^2}dx. \qquad (9.17)$$

This integral can be solved with a substitution, which we leave this as an exercise to the reader.

9.2 Applications

In this section we consider some applications of integration to problems in physics and engineering.

First, we consider the problem of computing the force that water applies on a dam (of trapezoidal shape). Recall first that the pressure that a force applies on a surface A is is given by the force per unit surface. So, negletting directions (forces are vectors!), we can write

$$P = \frac{F}{A}.$$

When we consider a fluid, and a surface A in the fluid at depth d, we find that $P = \rho g d$, because the force on the fluid is given by the column of water, which has weight $F = \rho g A d$ (volume times density). It is an experimental fact, that the pressure in afluid is independent of the direction considered. Therefore, if we change the orientation of the surface, as long as the difference in depth along the surface is negligible, we find that the pressure is the same.

Example 9.2.1. Consider a dam of trapezoidal shape, with sizes being 50 m for the top side, 30 m for the bottom side, and 20 m height. We want to find the force that water exerts on the dam, in the assumption that the water reaches 4 m below the top level of the dam.

From the previous discussion, we know how to compute the pressure as a function of depth. Knowing pressure would allow us to find the force by multiplying it by the area it is exerted on. However, since the pressure is not constant along the dam, we need to brake the problem in small portions of the dam in a way that we can compute the force piece by piece. We need to use small rectangles where the height is small, so that the depth has little variation, and in that case we can compute the force by considering the pressure as being constant.

We fix a coordinate system where the $\vec{x}$- axis is oriented vertically, and it has a zero at the surface of the water, and it is placed in the middle of the trapezoid. A small rectangle as described above is placed at distance x_i^* from the origin, which means that the average depth is x_i^*, and it has height Δx. We need to compute the area of the rectangle. To do so, observe that the total length is given by $w_i = 2(15 + a_i)$, where a_i is the lateral portion, which we need to compute. By considering similar triangles, we obtain for a_i:

$$\frac{a_i}{16 - x_i^*} = \frac{10}{20},$$

which gives

$$a_i = 8 - \frac{x_i^*}{2}.$$

We therefore have $w_i = 2(15 + a_i) = 46 - x_i^*$. Now, we can compute the force F_i on the portion of dam of area $w_i \Delta x$ as $F_i = P_i A_i$, where $P_i = \rho g x_i^*$, and $A_i = w_i \Delta x$. We get

$$F_i = \rho g x_i^* (46 - x_i^*) \Delta x.$$

To compute the force acting on the whole dam, we need to sum all forces F_i. So, $F_{\text{tot}} \approx \sum_{i=1}^n F_i = \sum_{i=1}^n \rho g x_i^* (46 - x_i^*) \Delta x$. To compute F_{tot} exactly, we need to perform the integral

$$F_{\text{tot}} = \rho g \int_0^{16} x(46 - x) dx.$$

We now consider the problem of finding the center of mass. This is the point of an object that we can use to create equilibrium. For instance, if we have a rod (of negligible mass) with two masses m_1 and m_2 attached to the ends of it, we can find a point in between the two masses such that if we place the rod over a fulcrum at exactly that point, the rod will be in equilibrium and will not rotate. Archimedes found the relation that the masses m_1 and m_2 have to satisfy with respect to the distances d_1 and d_2 from the point at which the fulcrum is placed in order to have equilibrium. This is

$$m_1 d_1 = m_2 d_2.$$

If we set a horizontal axis $\vec{x}$, and we have the point with mass m_1 closer to the zero of the axis than the other mass, we can express the distance d_1 from the fulcrum $\bar{x}$ as $d_1 = \bar{x} - x_1$, and $d_2 = x_2 - \bar{x}$, where x_1 and x_2 are the points where m_1 and m_2 are placed, respectively. Then, from $m_1 d_1 = m_2 d_2$ we find

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

The point $\bar{x}$ is called the center of mass of the system. The quantities $M_i = m_i x_i$ are called the moments of the mass m_i, so that the numerator in the expression for the position of the center of mass is the total moment of the system.

A similar approach can be used when we have several (say n) masses on a one dimensional rod. In that case we can simply get the center of mass as

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}.$$

Let us now consider the case where we want to compute the center of mass of a continuous object. Also, for the sake of simplicity, assume that this object is given by the space under a curve defined through a function $y = f(x)$. In this case, the discrete approach considered up to now does not work anymore. We need to consider a partition of the interval of the domain of the function, and then break the continuum into small blocks. Of course, these blocks are going to be rectangles.

Using some physical principles that involve the symmetries of objects, we assume that the center of mass of an object that is symmetric with respect to a line has to lie on the line. Therefore, for a rectangle, the symmetry of the system gives that the center of mass lies in the middle of the rectangle.

We are now in the position of using the approximation of small rectangles to compute the moments, and therefore the center of mass. When computing the moment M_y of the whole continuous object with respect to the $\vec{y}$-axis, we need to sum all the contribution of the rectangles R_i, where each moment $M_y(R)i$ is given by the product of mass (density times area) and distance of the center of the rectangle. We have

$$M_y(R_i) = \rho f(\bar{x}_i)\bar{x}_i \Delta x.$$

Summing these contributions together we get

$$M_y \approx \sum_{i=1}^{n} \rho f(\bar{x}_i)\bar{x}_i \Delta x.$$

Therefore, taking the limit of this procedure we obtain

$$\begin{aligned} M_y &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho f(\bar{x}_i)\bar{x}_i \Delta x \\ &= \int_a^b \rho x f(x) dx. \end{aligned}$$

A similar procedure now applies for the computation of M_x, where the distance of the center of R_i from the $\vec{x}$-axis is given by $\frac{1}{2} f(\bar{x}_i)$. So, we get

$$M_x(R_i) = \rho \frac{1}{2} f(\bar{x}_i)^2 \Delta x.$$

This gives us an appoximated moment

$$M_x \approx \sum_{i=1}^{n} \rho \frac{1}{2} f(\bar{x}_i)^2 \Delta x.$$

Finally, we get

$$\begin{aligned} M_x &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho \frac{1}{2} f(\bar{x}_i)^2 \Delta x \\ &= \frac{1}{2} \int_a^b \rho f(x)^2 dx. \end{aligned}$$

Once we have computed the moments, we can obtain the center of mass simply by dividing them by the total mass of the system. The total mass is obtained by multiplying density of the object and the total area. So, we get

$$\begin{aligned} m_{\text{tot}} &= \rho A \\ &= \int_a^b \rho f(x) dx. \end{aligned}$$

We now are in the poistion of writing the coordinates $(\bar{x}, \bar{y})$ of the center of mass, as

$$\begin{aligned} \bar{x} &= \frac{M_y}{m_{\text{tot}}} \\ &= \frac{\int_a^b \rho x f(x) dx}{\int_a^b \rho f(x) dx}, \end{aligned}$$

and

$$\begin{aligned} \bar{y} &= \frac{M_x}{m_{\text{tot}}} \\ &= \frac{\frac{1}{2}\int_a^b \rho f(x)^2 dx}{\int_a^b \rho f(x) dx}. \end{aligned}$$

Notice that while the density ρ is most easily assumed to be constant, we have not explicitly done so. In fact, our reasoning is applicable as long as ρ depends only on x and not on y, in other words, ρ depends only on the horizontal coordinate and not on the height. When ρ is constant, then the expression in $\bar{x}$ and $\bar{y}$ is does not depend on ρ, since we can just simplify the ρ at the numerator with the one at the denominator. In this case we can rewrite the formulas as

$$\bar{x} = \frac{\int_a^b x f(x) dx}{A},$$

and

$$\bar{y} = \frac{\frac{1}{2}\int_a^b f(x)^2 dx}{A}.$$

Exercise 9.2.2. Compute the center of mass of a half disk of radius r. (Hint: The x-coordinate is easy to compute.)

When dealing with an object that lies between two curves defined through the functions $y = f(x)$ and $y = g(x)$, the same porocedure as before gives the formulas

$$\bar{x} = \frac{\int_a^b x[f(x) - g(x)] dx}{A},$$

and

$$\bar{y} = \frac{\frac{1}{2}\int_a^b [f(x)^2 - g(x)^2]dx}{A}.$$

We now prove a very old theorem (due to the Greek mathematician Pappus).

Theorem 9.2.3 (Theorem of Pappus). *Let $\mathcal{R}$ be a plane region that lies completely on one side of a line l in the plane. Then, the volume V of the solid obtained by rotating $\mathcal{R}$ about l is given by*

$$V = Ad, \tag{9.18}$$

where A is the area of $\mathcal{R}$, and d is the distance traveled by the center of mass of $\mathcal{R}$ as $\mathcal{R}$ rotates about l.

Proof. We assume for the sake of simplicity that the plane region $\mathcal{R}$ is delimited by two functions $f(x)$ and $g(x)$, and that the roation is such that l coincides with the $\vec{y}$-axis.

We apply the method of cylindrical shells to the computation of V. We have

$$\begin{aligned} V &= 2\pi \int_a^b x[f(x) - g(x)]dx \\ &= 2\pi \bar{x} A \\ &= Ad, \end{aligned}$$

where we have empployed Equation (9.18) in the second equality, and $d = 2\pi\bar{x}$ is the distance traveled by the center of mass in the rotation. □

The previous result makes it particularly easy to compute, for instance, the volume of a torus (which is a fancy way of calling a doughnut). In fact, this can be obtained by rotating a disk of radius r about the $\vec{y}$ axis at a distance R. Then, the area of the disk is just $A = \pi r^2$. For reasons of symmetry, the center of the disk is the center of mass, and the distance traveled in a rotation with radius R is just given by $d = 2\pi R$. Applying Equation (9.18), we therefore obtain

$$\begin{aligned} V &= Ad \\ &= 2\pi^2 r^2 R. \end{aligned}$$

Chapter 10

Differential Equations

Differential equations (DEs) are equations involving an unknown function and its derivatives. The importance of DEs lies in the fact that they are so commonly employed to model real world problems. For instance, classical and quantum mechanics is described through differential equations. Models describing engineering problems are usually based on classical mechanics and involve DEs. Chemical dynamics and population dynamics in chemistry and biology, respectively, are based on DEs.

The use of DEs in describing real-world problems is due to the fact that they are equations that relate a quantity of interest (i.e. a function) and its rates of change. For instance, the relation between position and acceleration (which is the second derivative of position as a function of time), gives the law of motion (one of Newton's laws) in classical mechanics.

In this chapter we will introduce the concept of DE, and see how to solve some elementary equations. We will also develop some visual and numerical tools that help us understanding natural phenomena described through DEs.

10.1 Some motivating examples

We begin by considering some motivating examples. First, we consider the problem of modeling population growth.

10.1.1 Population growth

We want to model a given population (of animals, or bacteria, to say just two common examples). The fundamental assumption of such a model is that the population grows at a rate that is proportional to the size of the population. This is reasonable, when we do not consider external conditions such as potential lack of food (higher population means also higher amount of food needed), absence of predators and so on.

If time t is the independent variable, we call $P(t)$ the function that determines the population size. We are interested in undertanding how the population varies over time. Therefore, we need to understand how to relate P to its rate of growth. The latter, simply defines how the population changes within a certain time frame. Of course, we can express this as a derivative, since this would indicate the population variation in a very small time frame.

In the assumption that the population growth is proportional to the population, this means that

$$\frac{dP}{dt} = kP, \tag{10.1}$$

where k is the constant of proportionality. This constant depends on the type of population (e.g. certain types of animals will have one constant, bacteria will have another one), and in realistic models it will also depend on the environment (e.g. bacteria in one type of solution rather than another one).

Of course, we would like to understand what type of functions $P(t)$ would satisfy Equation (10.1). Such a solution to the equation would be a model of the population. We would need a function whose derivative is itself, up to a constant. We already know a funnction that satisfies such condition. In fact, the function $P(t) = e^{kt}$ has precisely this property. As an excercise, the reader should use $P(t) = e^{kt}$ and veerify that this satisfies the equation.

In general, if we take an arbitrary number c, and multiply the previous solution by it, we find that the equation is still satisfied. In fact, $P(t) = ce^{kt}$ is a solution to Equation (10.1) for any choice of c.

Our model has a free parameter, c, which we would have to understand in order to be able to model the population. To determine c, we need some information from the system (the population that we are modeling). For instance, if we knew $P(0)$, meaning the populaation at the first time of observation, which we call time zero, it would give us the value of c, since $P(0) = ce^{k\cdot 0} = ce^0 = c$.

Equation (10.1) has some serious limitations. Mostly, this is because it can model populations under some unrealistic assumptions, such as infinite amount of resources. This is not necessarily true. A more realistic type of behavior is that a population can grow exceptionally fast initially, but then it starts levelling off toward its *carrying capacity* M. Also, if for any reason it exceeds its capacity M, we would expect that it would decrease back toward it.

So, we need to modify our model (our DE) in such a way that these behaviors are taken into account. The previous observations can be incorporated into the DE by assuming that the growth is proportional not only to the population, but also the difference between carrying capacity and population. So, as P grows toward M, the difference $M - P$ becomes small, and the rate of growth decreases. Also, if P exceeds M, the difference $M - P$ will be negative, and the growth will be negative, so that the population decreases toward M again. Our model now reads

$$\frac{dP}{dt} = \hat{k}P(M - P) = kP(1 - \frac{P}{M}), \tag{10.2}$$

where $k = \hat{k}M$.

Equation 10.2 is called *logistic differential equation*. The solutions of this equation are not as straightforward to obtain as before, but one can see that they have the qualitative behavior of "convergin" toward the stable population $P = M$ over time. In other words, the solutions of the equation have an horizontal asymptote at $P = M$, showing that the population tends to become M, either by decreasing toward it, or increasing toward it.

10.1.2 Motion of Spring: Hooke's Law

Hooke's Law states that the force that a spring excerts over an object attached to it (say of mass m), is proportional to the stretch of the spring from its resting size. In other words, if we move the spring by its equilibrium position by x units, then we will have a force that is given by $F = -kx$, for some k that depends on the spring (manufacturers usually provide that). From Newton's Second Law, we also know that the force F is given by the product of mass m and the acceleration of the mass, which is given by $\frac{d^2x}{dt^2}$. So, we have the equation

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x. \tag{10.3}$$

The general solution of Equation (10.3) is a combination of sine and cosine functions, which is well aligned with our intuition, since we expect the mass to start oscillating around the equiilibrium position.

10.1.3 General differential equations

In general, a differential equation is an equation containing an unknown function and its derivatives. Finding a solution means that we find a function y such that the equation is satisfied when we plug y and its derivatives in the equation. In a sense, finding an indefinite integral is a form of solving a (very simple) differential equation, since we are substantially asked to solve $y' = f(x)$. In fact, if $y = g(x)$ is a function in $\int f(x)dx$, we have (by definition) that $y' = f(x)$. In a more proper way, these equations are called *ordinary differential equations*, or ODEs for short. Here ordinary means that there is only one independent variable, and derivatives do not refer to multiple variables (in which case one has partial derivatives, and *partial differential equations*, or PDEs).

In general, we can write the form of an ODE as

$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{10.4}$$

where F is some function that "mixes" x, y and higher derivatives of y up to the n^{th} derivative. We will consider some very special cases of functions F for which solving the corresponding ODE is very simple.

The *order* of a differential equation is the order of the highest derivative appearing in the equation. So, for instance, the order of the logistic differential equation is 1, while the order of the spring motion is 2.

As observed in the case of population growth, having a family of solutions means that we have some free parameter(s) that we would like to determine. So, usually we are interested in some specific solutions that satisfy some extra conditions. For instance, we want the value of y at the initial point t_0 to be a specified y_0. This means that we are picking the solution that passes through the point (t_0, y_0). This is called an *initial value problem*, or IVP. Intuitively, it corresponds to having the value of a given system at time t_0 (e.g. by experimental means), and predicting the behavior of the system at later times.

10.2 Direction fields and Euler's method

Generally speaking, it is extremely difficult to solve DEs but in some specific cases. However, one can use visual and numerical methods to understand the behavior of the system that is modeled through a DE even when an explicit solution is not known.

Direction fields allow us to draw solutions even when we do not know an explicit solution. For instance, consider the initial value problem

$$y' = x + y, \quad y(0) = 1 \tag{10.5}$$

. This means that if y is a solution, i.e. a curve in the plane that satisfies the equation, the slope of the graph of y needs to be equal to the sum of the x coordinate and the y coordinate of the point under consideration. Therefore, we can draw little line segments at points in the plane, where these lines represent the slopes of the graph of a solution $y(x)$. At the point $x = 0$, the initial value tells us that $y = 1$. Therefore, we have a diagonal little segment drawn at the point $(0, 1)$, of slope $y' = 0 + 1 = 1$.

Euler's method is a numerical approach to solve differential equations. We assume here to deal with an IVP of type

$$y' = F(x, y), \quad y(x_0) = y_0 \tag{10.6}$$

for some function F. The fundamental idea here is that we can numerically approximate the derivative $y' = \frac{dy}{dx}$ by considering very small fractions of type $\frac{\Delta y}{\Delta x}$, where Δx is the step of the Euler solver. Observe, that if we wanted to know $y(x_0 + \Delta x)$, we could think of the quotient from the defition of derivative as $\frac{\Delta y}{\Delta x} = \frac{y(x_0+\Delta x)-y(x_0)}{\Delta x}$. But, since this is an approximation to the derivative $\frac{dy}{dx}$, and the ODE gives us that $\frac{dy}{dx} = F(x, y)$, we can equate $\frac{y(x_0+\Delta x)-y(x_0)}{\Delta x}$ to $F(x_0, y_0)$. This gives us an estimate for $y(x_0 + \Delta x)$ as

$$y(x_0 + \Delta x) = y_0 + \Delta x F(x_0, y_0), \tag{10.7}$$

where we are using the fact that by the initial value, we have $y_0 = y(x_0)$. We set $y_1 = y(x_0 + \Delta x)$, and $x_1 = x_0 + \Delta x$. Now, we can apply the same procedure using x_1 and y_1 to obtain a subsequent point (x_2, y_2) as

$$y_2 = y_1 + \Delta x F(x_1, y_1). \tag{10.8}$$

Proceeding in this way for n times, we obtain the value of y_n at the point $x_0 + n\Delta x$ according to

$$y_n = y_{n-1} + \Delta x F(x_{n-1}, y_{n-1}). \tag{10.9}$$

We summarize this procedure in the following.

Method 10.2.1 (Euler's Method)**.** *Given an IVP as in Equation* (10.6)*, we can numerically solve it by choosing a step size Δx, and finding the value y_n at the points $x_n = x_0 + n\Delta x$ according to the numerical scheme*

$$y_n = y_{n-1} + \Delta x F(x_{n-1}, y_{n-1}). \tag{10.10}$$

Remark 10.2.2. Observe that we know how to perform the first step of the numerical scheme because the IVT gives us x_0 and y_0.

Let us go back to the example of the IVP (10.5). Choosing $\Delta x = 0.1$, and applying Method 10.2.1, we obtain Table 10.1.

n (step)	x_n	y_n
1	0.1	1.10000
2	0.2	1.220000
3	0.3	1.362000
4	0.4	1.528200
5	0.5	1.721020
6	0.6	1.943122
7	0.7	2.197434
8	0.8	2.487178
9	0.9	2.815895
10	1	3.187485

Table 10.1: Numerical solution of IVP (10.5) using Euler's Method.

10.3 Separable equations

Separable differential equations are ODEs of the first order that can be written in the form

$$\frac{dy}{dx} = g(x)f(y), \tag{10.11}$$

for some functions g and f. Separable here refers to the fact that we can separate the right hand side of Equation (10.11) in the product of a term containing only x, and another containing only y.

In the assumption that $f(y) \neq 0$, we can rewrite Equation (10.11) as

$$\frac{dy}{f(y)} = g(x)dx. \tag{10.12}$$

Then, a solution to Equation (10.11) can be obtained by integrating both sides of Equation (10.12):

$$\int \frac{dy}{f(y)} = \int g(x)dx. \tag{10.13}$$

The justification for this procedure comes from an application of the Chain Rule. In fact, if Equation (10.13) holds, by differentiating both sides of the equation with respect to x, and recalling that from the Chain Rule $\frac{d}{dx} = \frac{dy}{dx}\frac{d}{dy}$ we have

$$\frac{d}{dy}[\int \frac{dy}{f(y)}]\frac{dy}{dx} = \frac{d}{dx}\int g(x)dx$$

which gives

$$\frac{1}{f(y)}\frac{dy}{dx} = g(x),$$

which is equivalent to Equation (10.11).

Remark 10.3.1. After solving Equation (10.13), we find y in an implicit form. It is not necessarily true that we are going to be able to explicitly find y as a function of x.

We show the procedure with an example.

Example 10.3.2. Consider the equation

$$y' = x^2(3y+1). \tag{10.14}$$

After finding the general solution of this equation, we want to solve the IVP with $y(0) = -1$.

There is an immediate solution to Equation (10.14), which is obtained by taking the constant solution $y = -\frac{1}{3}$. In fact, in this case the equality reduces to an identity $0 = 0$. We need to find nonconstant solutions now.

The equation is separable, since the right hand side is a product of $g(x) = x^2$ and $f(y) = 3y+1$. We therefore rewrite it as

$$\frac{dy}{3y+1} = x^2 dx,$$

and integrate both sides obtaining

$$\int \frac{dy}{3y+1} = \int x^2 dx.$$

We now need to integrate both sides of the equation. We have $\int \frac{dy}{3y+1} = \frac{1}{3}\ln|3y+1| + \text{const}$ and $\int x^2 dx = \frac{1}{3}x^3 + \text{const}$. Using only a single integration constant c, we have found the equality

$$\ln|3y+1| = x^3 + k,$$

where $k = 3c$ is again just a simple constant. To find y explicitly (note that we have y only implicitly so far), now we need to exponentiate both sides of the previous equation, which gives us $|3y+1| = e^{x^3+k}$. We therefore have two possible solutions, $3y+1 = e^{x^3+k}$ or $3y+1 = -e^{x^3+k}$, which give us $y = \frac{1}{3}[e^{x^3+k} - 1]$ and $y = \frac{1}{3}[-e^{x^3+k} - 1]$, respectively.

Now, we have to select the solution (and the correct k) that satisfies the initial condition $y(0) = -1$. This gives us $-1 = \frac{1}{3}[e^k - 1]$ and $-1 = \frac{1}{3}[-e^k - 1]$. Observe that from the first one, we cannot find a solution, because $e^k > 0$, so that $\frac{1}{3}[e^k - 1] > -\frac{1}{3} > -1$. From the second equation, $-1 = \frac{1}{3}[-e^k - 1]$ we find the correct y by solving the equation. We obtain $-e^k = -2$, which gives us $e^k = 2$, and therefore $k = \ln 2$.

The solution to the IVP is therefore $y = \frac{1}{3}[-e^{x^3+\ln 2} - 1]$.

Example 10.3.3. We want to solve the differential equation $\frac{dy}{dx} = \frac{6x^2}{2y+\cos y}$.

This is a separable equation, since the right hand side can be written as a product $f(y)g(x)$ where $g(x) = 6x^2$ and $f(y) = \frac{1}{2y+\cos y}$. We can therefore obtain the solutions from Equation (10.13). We have

$$\int [2y + \cos y] dy \quad = \quad \int 6x^2 dx$$

which gives us

$$y^2 + \sin y \quad = \quad 2x^3 + k.$$

The solution relates y to x implicitly, and it is not clear in this case how to explicitly obtain y as a function of x.

We can now go back to the study of population growth and the logistic equation. We have found two equations that describe population growth over time. One was not very realistic, since it assumed a no restriction on food available to the population, no predators and so on. This was the differential equation

$$\frac{dP}{dt} = kP.$$

Of course, this is a separable equation, since $f(t) = k$ is a (constant) function of time, and $g(P) = P$ is a function that only contains P. Then we can find a general solution by using Equation (10.13). We find

$$\int \frac{dP}{P} = \int kdt,$$

which gives $\ln P = kt + c$. Taking exponents of both sides of the equation we find $P = e^{kt+c} = e^c e^{kt} = Ae^{kt}$, where we have set $A = e^c$. As already described at the beginning of the chapter, in order to find also the integration constant A, we need some extra information. For instance, if we know the value of the population at time zero $P(0)$, we will have $P(0) = Ae^0 = A$. Separation of variables has given us exactly the solution that we predicted before, based on intuition.

A more accurate model was then obtained by the logistic equation, which we did not solve. However, this is also a separable equation, and we can solve it now. Recall that the logistic equation is given by

$$\frac{dP}{dt} = kP(1 - \frac{P}{M}).$$

This is separable since the right hand side can be written as a product $g(t)f(P)$ where $g(t) = k$ (constant!), and $f(P) = P(1 - \frac{P}{M})$. Then we can obtain a solution by integrating the equation

$$\int \frac{dP}{P(1 - \frac{P}{M})} = \int kdt.$$

Clearly, we have $\int kdt = kt + c$. To integrate te left hand side, we use partial fractions. We have

$$\begin{aligned} \int \frac{dP}{P(1 - \frac{P}{M})} &= \int \frac{M}{P(M-P)} dP \\ &= \int [\frac{1}{P} + \frac{1}{M-P}] dP \\ &= \ln P - \ln |M - P| + k. \end{aligned}$$

Equating the two expressions that we have found, and renaming the constant of integration C we get the solution

$$\ln P - \ln |M - P| = kt + C,$$

which is an implicit form of the solution. We can obtain an explicit solution by rewriting $\ln P - \ln |M - P| = \ln \frac{P}{|M-P|} = -\ln \frac{|M-P|}{P}$ and then taking exponents of both sides of the equation, to get

$$\frac{|M - P|}{P} = \pm Ae^{-kt},$$

where we have set again $A = e^{-C}$. This gives us $\frac{M-P}{P} = \pm Ae^{-kt}$, where the sign depends on whether P is larger or smaller than M. Solving for P we get

$$P = \frac{M}{1 \pm Ae^{-kt}}. \tag{10.15}$$

This clearly shows that $\lim_{t\to\infty} P = M$, as claimed at the beginning of the chapter.

10.4 Linear Differential Equations

A linear differential equation is is a DE in the from

$$y' + g(x)y = f(x). \tag{10.16}$$

When $f(x) = 0$, the equation is said to be *homogeneous*, while when $f(x)$ is not zero, it is called *non-homogeneous*.

Before understanding how to solve these equations in general, we start with a simple example, and show how we can obtain a solution.

Example 10.4.1. Let us consider the equation $y' + \frac{1}{x}y = 1$. This is a non-homogeneous linear differential equation, where $g(x) = x$ and $f(x) = 1$.

Suppose $q(x)$ is a nonzero function. Then, we can multiply the whole equation by $q(x)$ and obtain the equation

$$q(x)y' + q(x)\frac{1}{x}y = q(x). \tag{10.17}$$

We start with the observation that the left hand side looks more or less like the derivative of a product. In fact, the derivative of a product of type $q(x)y$ is given by $\frac{d}{dx}(q(x)y) = q(x)y' + q'(x)y$, by using the Leibniz rule. So, if $q(x)$ is chosen in a way such that $q'(x) = q(x)\frac{1}{x}$, then in Equation (10.17) we would have precisely the derivative of $q(x)y$.

Now, the interesting thing is that $q'(x) = q(x)\frac{1}{x}$ is a separable differential equation, and we know how to solve such an equation. We have $q(x) = x + c$. Since we are interested in a function $q(x)$ that satisfies the equation $q'(x) = q(x)\frac{1}{x}$ and not the most general one, we can simply take $c = 0$. Using this $q(x)$ in Equation (10.17) we have obtained a new equation

$$\frac{d}{dx}(xy) = x. \tag{10.18}$$

We can integrate both sides of the equation to get

$$xy = \frac{1}{2}x^2 + k. \tag{10.19}$$

This gives us $y = \frac{1}{2}x + \frac{k}{x}$. Solving the equation was very simple in this case, but the same procedure can be seen to work more in general.

The general apporach is a generalization of the procedure unveiled in the previous example. In fact, given a linear DE as in Equation (10.16), we can multiply both sides of the equation by a function $q(x)$ which we want to determine in such a way to have

$$\frac{d}{dx}(q(x)y) = q(x)f(x).$$

Since $\frac{d}{dx}(q(x)y) = q(x)y' + q'(x)y$, this requires us to have $q'(x) = g(x)q(x)$. Of course, the latter is a separable differential equation which we can solve by separating the variables. A solution to the corresponding separable equation is given by $q(x) = e^{\int g(x)dx}$. We therefore obtain a function $q(x)$ so that our problem has now been reduced to solving $\frac{d}{dx}(q(x)y) = q(x)f(x)$. This can be solved by integrating both sides of the equation to obtain

$$q(x)y = \int q(x)f(x)dx, \tag{10.20}$$

from which we get a solution

$$y = \frac{1}{q(x)} \int q(x)f(x)dx. \tag{10.21}$$

The function $q(x)$ that allows us to solve the initial differential equation (10.16) is called an *integrating factor*. The procedure just described can be summarized in the following.

Method 10.4.2. *Let*

$$y' + g(x)y = f(x)$$

be a linear differential equation. To solve the equation, we can multiply both sides by the function $q(x) = e^{\int g(x)dx}$, *and obtain a solution as* $y = \frac{1}{q(x)} \int q(x)f(x)dx$.

Example 10.4.3. We want to solve the equation $y' + 3x^2y = 6x^2$.

The integrating factor is given by $q(x) = e^{\int 3x^2dx} = e^{x^3}$, where we have chosen the integration constant to be zero (as in Example 10.4.1). Having $q(x)$, we can now solve the original equation as

$$\begin{aligned} y(x) &= \frac{1}{e^{x^3}} \int 6x^2 e^{x^3} dx \\ &= \frac{1}{e^{x^3}} [2e^{x^3} + k] \\ &= 2 + ke^{-x^3}. \end{aligned}$$

Example 10.4.4. Let us consider another problem motivated by real world applications. Here we have a circuit with a battery that generates a time dependent voltage $E(t)$, with a corresponding electrical current $I(t)$. The voltage is measured in volts V, and the current in amperes A. Suppose that the circuit also has a resistor R measured in ohms Ω, and an inductor with inductance L measured in henries H.

The resistor and the inductor both induce a voltage drop, which can be written as $RI(t)$ (Ohm's Law), and $L\frac{dI}{dt}$, respectively. One of Kirchoff's laws states that the sum of the voltage drops equals the total voltage supplied $E(t)$. We therefore have a linear differential equation

$$L\frac{dI}{dt} + RI = E(t). \tag{10.22}$$

Let us consider now a battery that gives a constant voltage of $60V$, a resistance of 12Ω, and inductance of $4H$. Assume that at time zero we switch on the circuit, so that $I(0) = 0$. We want to compute the current $I(t)$ as a function of time, and find the limiting value of the current.

Our differential equation, after plugging the values given above in Equation 10.22 becomes

$$\frac{dI}{dt} + 3I = 15,$$

and with $I(0) = 0$ this is an IVP.

The integrating factor is obtained by solving $q(t) = e^{\int 3dt} = e^{3t}$, where as usual we are choosing a zero integration constant (recall that we just need one integrating function, rather than the most general one). Then, we have the equation

$$e^{3t}\frac{dI}{dt} + 3e^{3t}I = 15e^{3t},$$

from which we get

$$\frac{d}{dt}(e^{3t}I) = 15e^{3t}.$$

Integrating both sides of the equation we find $e^{3t}I = 15\int e^{3t}dt = 5e^{3t} + k$. Rewriting for $I(t)$ explicitly, we find

$$I(t) = 5 + ke^{-3t}.$$

To find the limiting value of $I(t)$ we need to take the limit $t \to \infty$. Since $e^{-3t} \to 0$, as $t \to \infty$, it follows that the limiting value is 5 amperes.

Example 10.4.5. This is a qualitative example related to biology. This model, called the Lotka-Volterra equations, or Predator-Prey system, is a good model for the population dynamics relating predators and prey in a biological environment.

We indicate by $R(t)$ the number of preys in an enviroment as a function of time, while $W(t)$ indicates the number of predators in the same environment. A good example might be the poopulation of wolves and rabbits. We know that in an enviroment that is ideal, a population can grow exponentially. For wolves, we assume that we have an exponential decay, since without preys, the wolves would not be able to feed temselves.

The population of preys diminishes with a rate proportional to the predators, since the more the predators, the more the preys are being hunted. The number of encounters between the two species depends on the size of the two poopulations. So, the equation for the preys is given by $\frac{dR}{dt} = kR - aRW$. Similarly, for the predators we get $\frac{dW}{dt} = -rW + bRW$. Here, k, r, a, b are all constants that depend on the populations under consideration.

The model that describes the population dynamics is given by

$$\begin{cases} \frac{dR}{dt} = kR - aRW \\ \frac{dW}{dt} = -rW + bRW. \end{cases} \tag{10.23}$$

This is a system of two linear differential equations whose solution gives the variation of the two populations with respect to time. A typical solution of the Lotka-Volterra equations shows an oscillatory behavior (with shifted oscillations between the two populations). We have not learned how to deal with systems, so we will not consider this system in detail, but it shows yet another application of the study of differential equations in practical problems.

Chapter 11

Sequences, Series, and Power Series

The notions of sequence and series are extremely important in mathematical analysis, and more generally in all mathematics. Before delving into the subject, we give a motivating example from differential equations.

We consider the differential equation

$$y'' + y = 0. \tag{11.1}$$

We would like to find a general solution to this equation. Of course, we can immediately see that $\sin x$ and $\cos x$ are both solutions of the equation by direct inspection. However, we do not know if this is a general result. Moreover, how should we proceed when the solution is not so obvious? So, even if this problem is very simple, it is worth having a look at it from a different perspective.

We now proceed in a heuristic way, and we will formalize several concepts that we consider now in this chapter. We assume that we have a test function $y(x) = \sum_{n=0} a_n x^n$ which is a polynomial of some degree, where the coefficients a_n are just numbers. We do not concern ourselves excessively with writing the extremes of summation and the degree of the polynomial, as we are proceeding in an intuitive way here. It is clear that in order to completely determine $y(x)$, we would have to determine the coefficients a_n. But how do we do that? We could imagine to insert our test function in Equation (11.1) and see if the equation somehow forces the coefficients a_n to have some specific values. That would do the job.

We have $y'(x) = \sum_{n=1} n a_n x^{n-1}$, where we have used the power rule. To compute $y''(x)$, we can again use the power rule, and obtain $y''(x) = \sum_{n=2} n(n-1) a_n x^{n-2}$. Putting everything together, we need to have the new equation

$$\sum_{n=2} n(n-1) a_n x^{n-2} + \sum_{n=0} a_n x^n = 0. \tag{11.2}$$

Now we can inspect directly what we need to have in order to obtain that Equation (11.2) is satisfied. First, we would need to have the coefficients for the term x^0 to be the same. The first term gives x^0 when $n = 2$, while the second term gives zero when $n = 0$. So, we have an equality $2a_2 + a_0 = 0$. Then, we have to inspect what happens with the coefficients of x^1. This means that in the first term, $n = 3$, while the second term has $n = 1$. We get an equality $3 \cdot 2 \cdot a_3 + a_1 = 0$. More generally, terms x^k come from setting $n = k + 2$ in the first term, and $n = k$ in the second term. So, we need the equality

$$(n+2)(n+1) a_{n+2} + a_n = 0. \tag{11.3}$$

This gives us the recurrence relation $a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$.

In other words, if a function $y(x)$ written as a polynomial is a solution of Equation (11.1), its coefficients have to satisfy the relation in Equation (11.3). Now we let ourselves wonder for a minute. It would be great if any function could be a polynomial, because it would mean that our solution is general. Of course, we know too well that this is not the case. However, there is something very nice that is quite similar to this. While not every function is a polynomial, it happens that regular enough functions can be written as an infinite polynomial (basically where the terms never stop). It turns out that the same reasoning above will be applicable in this case and we can find the solutions of DEs following the same procedure.

To conclude, from Equation (11.3) we find that when n is even, i.e. when $n = 2k$, we can write $a_{2k} = a_0\frac{(-1)^k}{(2k)!}$, while when n is odd, i.e. when $n = 2k+1$, we have $a_{2k+1} = a_1\frac{(-1)^k}{(2k+1)!}$. It turns out (and we will see it in this chapter), that these are the coefficients for cosine and sine functions, respectively. Therefore, we have found that a general solution (which is regular enough) to Equation (11.1) is a linear combination of sine and cosine. So, the solutions that we discussed above were quite accurate, after all.

11.1 Sequences

A sequence is a list of intinitely many real numbers $a_1, a_2, \dots, a_n, \dots$ indexed by the natural numbers $n = 0, 1, 2, \dots$. More formally, one can think of a sequence as a function $a : \mathbb{N} \longrightarrow \mathbb{R}$ that takes a natural number n, and outputs a real number $a(n)$, where we use n as a subscript as a_n for simplicity of notation.

Any function that we have already encountered, as long as the numbers $n = 0, 1, \dots$ are part of its domain, defines a sequence, simply by restricting the domain to $\mathbb{N}$. For instance, consider the function $f(x) = x^2$. Then, we can consider the sequence of numbers $a_n = n^2$, which is obtained by applying $f(x)$ only on the numbers $n = 0, 1 \dots$.

The notation employed to indicate sequences is $\{a_n\}$ or sometimes $\{a_n\}_{n=0}^{\infty}$. Generally speaking, a sequence can be defined by giving a formula for its n^{th} term. The convenience of the notation $\{a_n\}_{n=0}^{\infty}$ is that it shows the "first" term of a_n explicitly. In fact, it is often useful to define a_n with $n \geq d$ for some number d. For instance, consider the sequence $a_n = \sqrt{n-2}$. Of course, for $n = 0, 1$ this sequence would not make any sense in the real numbers, so that we need to specify that the first n is 2 by saying that $n \geq 2$. Concisely, we can write $\{\sqrt{n-2}\}_{n=2}^{\infty}$.

The following example shows several sequences.

Example 11.1.1. The following are examples of sequences. The reader is invited to compute the first few terms of each of them to understand how the definition works.

- $a_n = \frac{1}{2^n}$.
- $a_n = \frac{n}{n+1}$.
- $a_n = e^{\frac{n}{n+1}}$.
- $a_n = \sqrt{n+1}$.

Another procedure to define a sequence is by giving a recurrence formula. Recurrence formulas are formulas where we are able to obtain the n^{th} term by knowledge of the previous terms. We show this procedure with a famous example: The Fibonacci sequence.

Example 11.1.2. We indicate this sequence by $\{f_n\}$ since it is named after Fibonacci. This is defined by the following assignment.

- $f_0 = 0$.
- $f_1 = 1$.
- $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$.

The meaning of this definition is the following. We know how to get f_0 and f_1, because they are given to us. To obtain f_2, we need to set $n = 2$ in the formula $f_n = f_{n-1} + f_{n-2}$. This means that $f_2 = f_1 + f_0 = 1 + 0 = 1$. To obtain f_3, we set $n = 3$ in $f_n = f_{n-1} + f_{n-2}$, therefore obtaining $f_3 = f_2 + f_1 = 1 + 1 = 2$. In this way we can obtain any term of f_n, by knowing the preceding terms.

When we considered functions, our main interest was in limits, through which we were able to compute derivatives, and also define the notion of continuity. However, observe that finite limits for sequences do not make any sense. In fact, the points of the domain of a sequence are isolated since there are gaps between 0 and 1, or 1 and 2 and so on. There is one notion of limit that makes sense for sequences, and it is of great importance. This is the limit of a_n as n goes to ∞, which is conceptually similar to the notion of the limit of $f(x)$ as x goes to ∞.

We say that a sequence $\{a_n\}$ has the (finite) limit L if as n grows, a_n gets closer and closer to L, possibly without never reaching it. The limit of a sequence, if it exists, is indicated by the symbol

$$\lim_{n\to\infty} a_n = L.$$

Since it is clear that $n \to \infty$ (see discussion above), sometimes we simply omit it and write

$$\lim_{n} a_n = L,$$

$$\lim a_n = L,$$

or also

$$a_n \to L.$$

In this case we say that $\{a_n\}$ converges to L, and we say that the sequence has a convergent behavior.

Example 11.1.3. Consider the sequence $a_n = \frac{1}{n}$, where $n \geq 1$. Then, as n grows and takes larger and larger values, e.g. $n = 10, 100, 1000, 10000$, and so on, the fraction $\frac{1}{n}$ becomes smaller and smaller. In fact, we have that $\lim a_n = 0$.

Observe that in this example a_n never reaches 0, but it gets closer and closer to it.

A proper way of formalizing the intuitive idea of convergence explained above is by the following definition.

Definition 11.1.4. A sequence $\{a_n\}$ is said to converge to the number L, or it is said to have limit L, if for any choice of $\epsilon > 0$, we can find a natural number ν such that for all $n > \nu$ we have

$$|a_n - L| < \epsilon.$$

This is indicated by either of the symbols

$$\lim_{n\to\infty} a_n = L, \quad \lim_n a_n = L, \quad \lim a_n = L, \quad a_n \to L.$$

Similarly, we can define the limit of a_n when the function does not get closer and clsoer to a number, but its values increase without any bound and a_n goes to ∞. We pose the following definition.

Definition 11.1.5. The sequence $\{a_n\}$ is said to be divergent to ∞ if for any positive number $M > 0$ we can find a natural number ν such that whenever $n > \nu$ we have

$$a_n > M.$$

We indicate this situation by either of the symbols

$$\lim_{n\to\infty} a_n = \infty, \quad \lim_n a_n = \infty, \quad \lim a_n = \infty, \quad a_n \to \infty.$$

A similar definition can be posed for limits that go to $-\infty$. Such a sequence is said to be divergent.

We have the following extremely useful result to determine the limit of sequences.

Theorem 11.1.6. *Suppose that the sequence a_n is derived from a function $f(x)$ as $f(n) = a_n$. Then,*

- *If $\lim_{x\to\infty} f(x) = L$, we have $a_n \to L$.*
- *If $\lim_{x\to\infty} f(x) = \infty$, we have $a_n \to \infty$.*

Theorem 11.1.6 is particularly useful because it allows us to use tools from Calculus I such as de L'Hôpital's rule to compute limits of sequences.

Example 11.1.7. We want to compute the limit $\lim \frac{\ln n}{n}$, of the sequence $a_n = \frac{\ln n}{n}$.

We set $f(x) = \frac{\ln x}{x}$, with $x > 0$, which is continuous in its domain. Observe that $a_n = f(n)$. So, if we are able to compute the limit $\lim_{x\to\infty} f(x)$, it follows that a_n will have the same limit. To compute $\lim_{x\to\infty} f(x)$ we can use de L'Hôpital's rule. We have

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{\ln x}{x} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} \\
&= 0.
\end{aligned}$$

It therefore follows, applying Theorem 11.1.6, that $\lim \frac{\ln n}{n} = 0$ as well.

We now list some properties of convergent sequences.

Proposition 11.1.8. *Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences, and let c be a constant. Then, the following results hold.*

(i) $\lim(a_n + b_n) = \lim a_n + \lim b_n$.

(ii) $\lim(a_n - b_n) = \lim a_n - \lim b_n$.

(iii) $\lim ca_n = c \lim a_n$.

(iv) $\lim a_n b_n = \lim a_n \cdot \lim b_n$.

(v) $\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$, *if* $\lim b_n \neq 0$, *and* $b_n \neq 0$ *for all* n *large enough (so that both left hand side and right hand side make sense).*

(vi) $\lim a_n^p = [\lim a_n]^p$ *for* $p > 0$ *and* $a_n > 0$.

Proposition 11.1.9. *(i) If* $\lim a_n = \lim b_n = \infty$, *then* $\lim(a_n + b_n) = \infty$.

(ii) If $\lim a_n = \infty$ *and* $\lim b_n = -\infty$, *then* $\lim(a_n - b_n) = \infty$.

(iii) If $\lim a_n = \infty$, *and* $\lim b_n = L \neq 0$, *then*

$$\lim a_n b_n = \begin{cases} \infty & \text{if } L > 0 \\ -\infty & \text{if } L < 0 \end{cases}.$$

The Squeeze Theorem also holds in the case of sequences.

Theorem 11.1.10 (Squeeze Theorem)**.** *Assume that there exists a natural number* ν *such that* $a_n \leq b_n \leq c_n$ *for all* $n > \nu$. *Then, if* $\lim a_n = \lim c_n = L$, *it follows that* $\lim b_n = L$.

In other words, if at some point, when n becomes large enough, i.e. it is larger than some number ν, the values of b_n are always between a_n and c_n, if b_n is forced to have the same limit of a_n and c_n, if they converge to the same value. The Squeeze Theorem is very useful to show that certain sequences are converging. This is a very powerful tool.

Observe that similarly to the case of functions here we need $\lim a_n = \lim c_n$ to be able to say that $\lim b_n$ is L. If $\lim a_n = L_1$ and $\lim c_n = L_2$ and they are different, in general we can only say that $L_1 \leq \lim b_n \leq \lim c_n$, if this limit exists. In fact, we can very well have that such limit does not exist.

Example 11.1.11. Consider the sequences $a_n = -5$ and $c_n = 5$ for all n. Let $b_n = (-1)^n$ be the sequence that alternates between -1 and 1. Clearly, b_n does not converge, since it will always bounce between -1 and 1 without ever stabilizing. Also, since a_n and c_n are constant, they are convergent. But we have also that $a_n \leq b_n \leq c_n$ for all n. What went wrong here, is that $\lim a_n \neq \lim c_n$.

Theorem 11.1.12. *If* $\lim |a_n| = 0$, *then it follows that* $\lim a_n = 0$.

Let us now consider some examples of convergent and divergent sequences.

Example 11.1.13. We want to determine whether the sequence $a_n = \frac{n}{n+1}$ is convergent or divergent.

We proceed in a way that is similar to how we dealt with limits of functions, when $x \to \infty$. We group n and we rewrite

$$\begin{aligned}
\lim \frac{n}{n+1} &= \lim \frac{n}{n} \frac{1}{1+\frac{1}{n}} \\
&= \lim \frac{1}{1+\frac{1}{n}} \\
&= \frac{\lim 1}{\lim 1 + \lim \frac{1}{n}} \\
&= \frac{1}{1+0} \\
&= 1.
\end{aligned}$$

where we have used Proposition 11.1.8, and the limit $\lim \frac{1}{n} = 0$, since as n keeps growing, $1/n$ becomes smaller and smaller without ever becoming smaller than zero.

Generally speaking, whenever a sequence is defined as a fraction of polynomials in n, we can always group the largest power from numerator and denominator, and then take the limits of all the terms.

Example 11.1.14. Let $a_n = \frac{n^3-2n+1}{2n^2+1}$. We want to determine whether the sequence is convergent or divergent.

We group the largest powers both from numerator and denominator. We have

$$\begin{aligned}
\lim \frac{n^3-2n+1}{2n^2+1} &= \lim \frac{n^3}{n^2} \frac{1-2\frac{1}{n^2}+\frac{1}{n^3}}{2+\frac{1}{n^2}} \\
&= \lim n \frac{1-2\frac{1}{n^2}+\frac{1}{n^3}}{2+\frac{1}{n^2}}.
\end{aligned}$$

As computed before, using Proposition 11.1.8, and the fact that $\frac{1}{n^3}$ and $\frac{1}{n^2}$ converge to 0, we find that $\lim \frac{1-2\frac{1}{n^2}+\frac{1}{n^3}}{2+\frac{1}{n^2}} = \frac{1}{2}$. Of course, we also have that $\lim n = \infty$. So, we can apply Proposition 11.1.9 (iii) with a_n divergent and b_n convergent to $L = \frac{1}{2} > 0$, and we obtain that $\lim \frac{n^3-2n+1}{2n^2+1} = \infty$.

Example 11.1.15. Let $a_n = \frac{(-1)^n}{n}$. In this case, to evaluate the limit (and to determine if it exists or not), we can use the absolute value of a_n and apply Theorem 11.1.12.

In fact, we have

$$\begin{aligned}
\lim |\frac{(-1)^n}{n}| &= \lim \frac{1}{n} \\
&= 0.
\end{aligned}$$

So, Theorem 11.1.12 gives us that $\lim a_n = 0$ as well.

Theorem 11.1.16. *If $\lim a_n = L$, and the function $f(x)$ is continuous at L, then we have*

$$\lim f(a_n) = f(L).$$

Example 11.1.17. Consider the sequence $a_n = \frac{1}{n}$. Let $f(x) = e^x$. We have already seen that $\frac{1}{n} \longrightarrow 0$. So, applying Theorem 11.1.16 it follows that $f(\frac{1}{n}) = e^{\frac{1}{n}} \longrightarrow e^0 = 1$.

We now consider an example where the use of the Squeeze Theorem simplifies the problem greatly.

Example 11.1.18. Set $a_n = \frac{n!}{n^n}$. We want to determine whether this sequence is convergent or divergent (or if it does not have a limit).

If we write some of the terms of this sequence, we see that

- $a_1 = 1$.
- $a_2 = \frac{1 \cdot 2}{2 \cdot 2}$.
- $a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$.
- In general we have $a_n = \frac{1 \cdot 2 \cdots (n-1) \cdot n}{n \cdot n \cdots n \cdot n}$, where the denominator has n multiplied by itself n times.

The general way to write a_n shows that we can write

$$a_n = \frac{1}{n} \frac{2 \cdot 3 \cdots (n-1) \cdot n}{n \cdot n \cdots n \cdot n} \tag{11.4}$$

$$\leq \frac{1}{n} \frac{n \cdot n \cdots n \cdot n}{n \cdot n \cdots n \cdot n} \tag{11.5}$$

$$= \frac{1}{n}. \tag{11.6}$$

Therefore, we have $0 < a_n \leq \frac{1}{n}$. since the constant sequence 0 converges to 0, and $\frac{1}{n} \longrightarrow 0$, the Squeeze Theorem gives us that $a_n \longrightarrow 0$ as well.

Example 11.1.19. We consider the sequence $a_n = r^n$, where r is a fixed number. We want to determine for what values of r this sequence converges.

In this case, Calculus I helps us again, with the use of Theorem 11.1.6. In fact, we know that the function $f(x) = r^x$ has limits

$$\lim_{x \to \infty} r^x = \begin{cases} \infty & r > 1 \\ 0 & 0 < r < 1 \end{cases} .$$

It follows that a_n diverges for $r > 1$ and converges for $r < 1$. Of course, $1^n = 1$ for all n, and $0^n = 0$ for all n. Therefore, $a_n \to 0$ for $r = 0$, and $a_n \to 1$ for $r = 1$.

When $-1 < r < 0$, we have that $0 < |r| < 1$, and therefore from the previous considerations $|r^n| = |r|^n \longrightarrow 0$. Applying Theorem 11.1.12 we obtain that $a_n \longrightarrow 0$ as well. When $r \leq -1$, the sequence alternates between positive and negative numbers, and the absolute value does not converge. Therefore, the limit does not exist.

The resut is summarized below:

$$\lim r^n = \begin{cases} 0 & -1 < r < 1 \\ 1 & r = 1 \\ \infty & r > 1 \\ \text{DNE} & r < -1 \end{cases} .$$

We now introduce a concept of great importance. Namely, we will discuss the notion of *monotonic* sequence.

Definition 11.1.20. A sequence $\{a_n\}$ is said to be *increasing* if $a_n \leq a_{n+1}$ for all n. If $a_n < a_{n+1}$ then we say that $\{a_n\}$ is *stricly increasing*. Similarly, a function is said to be *decreasing* if $a_n \geq a_{n+1}$ for all n. If $a_n > a_{n+1}$, then we say that the sequence is *strictly decreasing*.

A sequence that is either (strictly) increasing or (strictly) decreasing is said to be (strictly) monotonic.

Remark 11.1.21. There are sequences that do not satisfy $a_n \leq a_{n+1}$ or $a_n \geq a_{n+1}$ for all n, but they satisfy this property for n that is large enough. These sequences are said to be *eventually monotonic* (with the same convention as above regarding increasing and decreaasing nomenclature).

Example 11.1.22. Consider the sequence $a_n = \frac{1}{n}$, defined for $n \geq 1$.

Since $\frac{1}{n} \geq \frac{1}{n+1}$ for all $n \geq 1$, it follows that this sequence is decreasing. In fact, it is immediate to verify that this sequence is strictly decreasing.

Example 11.1.23. We now want to show that the sequence $a_n = \frac{n}{n^2+1}$ is decreasing (with $n \geq 1$).

We have to check that $a_n \geq a_{n+1}$. Using the definition of a_n, this means that we have to verify that

$$\frac{n}{n^2+1} \geq \frac{n+1}{(n+1)^2+1},$$

which gives us (upon clearing denominators)

$$n[(n+1)^2+1] \geq (n+1)(n^2+1).$$

The previous inequality can be rewritten as $n^2+n \geq 1$. Since $n \geq 1$, n^2+n is always greater than 1. Therefore, the sequence is decreasing. In fact, since the inequality $n^2+n > 1$ holds with strict sign, it follows that the sequence is actually strictly decreasing.

Definition 11.1.24. A sequence is said to be *bounded above* if we can find a number M such that $a_n \leq M$ for all n. A sequence is said to be *bounded below* if we can find a number m such that $a_n \geq m$ for all n.

A sequence that is bounded both from below and above, is simply said to be *bounded*.

Example 11.1.25. The sequence $a_n = \frac{1}{n}$ is bounded above by 1, and below by 0. Therefore, it is bounded.

The sequence $a_n = n$ is bounded below by 0, but it is not bounded above.

The sequence $a_n = -n$ is bounded above by 0, but not bounded below.

Definition 11.1.26. Let $A \subset \mathbb{R}$ be a subset of $\mathbb{R}$. We say that b is an *upper bound* for A if $x \leq b$ for all x in A. Similarly, we say that c is a *lower bound* for A if $x \geq c$ for all x in A.

A *least upper bound* or *supremum* for the set A, denoted by $\sup A$, is an upper bound such that any number smaller than $\sup A$ is not an upper bound. A *greatest lower bound* or *infimum* for a set A, denoted by $\inf A$, is a lower bound such that any number larger than $\inf A$ is not a lower bound.

By definition of $\sup A$, is that if we consider a number $q = \sup A - \epsilon$, where $\epsilon > 0$ is arbitrarily small, we have that q is not an upper bound. This means that there must exist a number a in the set A which is greater than q. The *Completeness Axiom* states that any nonempty subset of the real numbers A that has an upper bound, has also a supremum. Similarly, any nonempty subset that has a lower bound, must also have an infimum.

Example 11.1.27. Consider $A = [0, 1]$. Then, in this case, $\sup A = 1$ and $\inf A = 0$. Observe, however, that the supremum and infimum do not need to be part of a set. In fact, $\sup[0, 1) = 1$ as well. This is because if we consider a number larger than 1, say 1.001, we can find infinitely many numbers between 1 and 1.001 that are not in the set $[0, 1)$. So, the only number that is "glued" from above to $[0, 1)$ is 1, even when the interval does not contain 1.

Example 11.1.28. Consider the set $A = \{\frac{1}{n}\}$ which is determined by the sequence $a_n = \frac{1}{n}$. Then, $\sup A = 1$, while $\inf A = 0$. For the supremum, this is relatively obvious, since we have a largest element, namely 1 of A. For the infimum, $\inf A = 0$ follows from the fact that $\frac{1}{n} \longrightarrow 0$, so if we choose any number slightly above 0, we can find infinitely many numbers from the sequence $\frac{1}{n}$ that are above 0 but are smaller than this chosen number!

Theorem 11.1.29. *The following facts hold.*

- *Any monotonic sequence has a limit.*
- *Any monotonic bounded sequence is convergent.*

In other words, a monotonic sequence is either divergent to $\pm\infty$ or convergent to a finite number, and this depends on whether it is bounded or not.

Proof. Let $\{a_n\}$ be a monotonic sequence. For the sake of clarity, assume that it is increasing. The case of a decreasing sequence can be treated analogously. Suppose that $\{a_n\}$ is not bounded above. Then, for any choice of $M > 0$, there must exist a natural number ν such that $a_\nu > M$, or otherwise a_n would be bounded above by M. Since a_n is increasing, for all $n > \nu$, we have that $a_n \geq a_\nu > M$. Since M was arbitrary, it means that $\lim a_n = \infty$. Now, suppose that a_n is bounded above. Then, we can find an M such that $a_n \leq M$ for all n. This means that the set $\{a_n\}$ has an upper bound. We claim now that $L := \sup\{a_n\} = \lim a_n$. For any choice of $\epsilon > 0$, we have that $L - \epsilon$ is not an upper bound, which means we can find some natural number ν such that $L \geq a_\nu > L - \epsilon$, and therefore $|L - a_\nu| < \epsilon$. However, since a_n is increasing, it follows that $L \geq a_n \geq a_\nu$ for all $n > \nu$. It follows that $a_n \longrightarrow L$. This completes the proof. □

11.2 Series

Series are the limits of a special type of sequence, obtained by summing all the terms of a given sequence. They formalize the notion of summing infinitely many terms together. This has been encoutnered, for example, when we defined integrals!

Definition 11.2.1. Let $\{a_n\}$ be a sequence. We define the sequence of *partial sums* $s_n = \sum_{i=0}^{n} a_i$ as the sequence consisting of the sum of the first n terms of $\{a_n\}$. A series, indicated as an infinite sum as $\sum_{i=0}^{\infty} a_i$, is the limit of the sequence of partial sums, if it exists. If the limit is finite, we say that the sequence is convergent. If the limit exists but it is infinite, we say that the series diverges. If the limit does not exist, we say that the series is not defined. In symbols, we have

$$\sum_{i=0}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=0}^{n} a_i.$$

Sometimes we will shorten notation to $\sum a_i$ to indicate a series. To simplify notation, occasionally we also say that the series diverges when its partial sums do not have a limit (this is also called oscillatory divergence sometimes).

Remark 11.2.2. As for the case of sequences, series can start at any arbitrary point. So, rather than $\sum_{i=0}^{\infty} a_n$, we can also encounter $\sum_{i=1}^{\infty} a_n$, or $\sum_{i=7}^{\infty} a_n$.

Example 11.2.3. Consider the series $\sum_{i=1}^{\infty} \frac{1}{n(n+1)}$. We want to determine whether the series is convergent, or divergent. If convergent, we want to compute the sum of it.

To do so, we need to determine the partial sums, and then take the limit of this. In other words, we need to compute

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}.$$

Observe that

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1},$$

simply by taking a common fraction of the right hand side. Therefore, the partial sums s_n can be written as

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{1}{i(i+1)} \\ &= \sum_{i=1}^{n} [\frac{1}{i} - \frac{1}{i+1}] \\ &= [1 - \frac{1}{2}] + [\frac{1}{2} - \frac{1}{3}] + [\frac{1}{3} - \frac{1}{4}] \cdots + [\frac{1}{n} - \frac{1}{n+1}] \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

We are therefore in the position of computing the limit of s_n, and this will determine whether the series is convergent or divergent (or if it is not defined). We now have

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{1}{n(n+1)} &= \lim s_n \\ &= \lim[1 - \frac{1}{n+1}] \\ &= 1 - 0 \\ &= 1. \end{aligned}$$

The series is convergent.

The series in Example 11.2.3 is a well known case of *Telescoping Series*. These series are characterized by the fact that their terms can be written as differences of consecutive terms of another sequence. In general, the series $\sum_{i=0}^{\infty} a_n$ is a telescoping series if we can find a sequence b_n such that $a_n = b_{n+1} - b_n$. In the example, we had $b_n = -\frac{1}{n}$. Since each term is a difference, the cancellation of the consecutive terms will happen exactly as in the example, and we can therefore compute the series directly as a limit of b_n.

Exercise 11.2.4. Let $\sum_{i=0}^{\infty} a_n$ be a telescoping series where $a_n = b_{n+1} - b_n$ for some sequence b_n satisfying $\lim b_n = 0$. Show that the series is convergent, and that its infinite sum is given by $\sum_{i=0}^{\infty} a_n = -b_0$. Hint: repeat what has been done in Example 11.2.3 step by step.

Geometric Series

The geometric series is of particular importance in mathematics and applications. Recall that the geometric sequence is defined by $a_n = r^n$ where r is some number. We have characterized when a_n converges and diverges. The geometric series is the sum of the elements a_n of the geometric sequence: $\sum_{n=0}^{\infty} ar^n$, where a and r are fixed numbers.

First, suppose that $r = 1$. Then, the partial sums of the series are given by $s_n = \sum_{k=0}^{n} a1^k = a + \cdots + a = na \to \pm\infty$, where the plus or minus sign depend on whether a is positive or negative. So, in this case the series is divergent. Let us now assume that $r \neq 1$. In this case, we have that $s_n = a + ar + ar^2 + \cdots + ar^n$, and $rs_n = ar + ar^2 + ar^3 + \cdots + ar^n + ar^{n+1}$. By subtracting both equations we obtain

$$s_n - rs_n = a - ar^n,$$

from which we get

$$s_n = a\frac{1 - r^n}{1 - r}. \tag{11.7}$$

We now need to compute the limit of s_n, and this will give us the result. We know that when $-1 < r < 1$, $r^n \to 0$. So, for such values of r we have

$$\lim s_n = \lim a\frac{1 - r^n}{1 - r} = \frac{a}{1 - r}.$$

This means that when $|r| < 1$, i.e. when $-1 < r < 1$, the geometric series converges to $\frac{1}{1-r}$. When $r > 1$ one can see that $\lim s_n = \infty$, and the series diverges, while for $r \leq 1$ the limit does not exist.

Test for divergence

We first show an interesting example, and then give a general criterion for divergence.

Example 11.2.5. Let us consider the *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$. We want to show that the series is divergent.

We consider only the terms of partial sums with indices that are powers of 2: $s_2, s_4, s_8, s_{16}, s_{32}, s_{64}$ and so on. We can concisely write them as s_{2^k}. If we show that these terms become larger and larger, it means that the series cannot converge, since convergence would mean that s_n has a limit, and this is a property that is defined by a statement of type "for all n larger than...", while we are showing that there are always terms that grow unboundedly. Let us look at the terms s_{2^k}.

$$\begin{aligned} s_2 &= 1+\frac{1}{2} \\ s_4 &= 1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4}) > 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4} = 1+\frac{2}{2} \\ s_8 &= 1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}) > 1+(\frac{1}{4}+\frac{1}{4})+(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}) = 1+\frac{3}{2}. \end{aligned}$$

One can proceed analogously for all terms s_{2^k} and see that there is an inequality of type $s_{s^n} > 1+\frac{n}{2}$. Since $1+\frac{n}{2} \longrightarrow \infty$ as n grows, we see that the terms of type s_{2^n} grow to infinity, which means that s_n cannot converge, since some of its terms go to infinity. One can show that $\lim s_n = \infty$ by using Theorem 11.1.29, since the partial sums are monotonic increasing (all terms are positive, so the partial sums become bigger), and the sequence of partial sums cannot be bounded as we have shown, since s_{2^n} goes to infinity. Therefore, the limit of the paratial sums must be ∞, and the series diverges.

The following result gives us a direct criterion to show that a series is divergent. It formalizes the thought that if the series converges, the terms that are being added must become increasingly small.

Theorem 11.2.6. *If the series $\sum_{n=0}^{\infty} a_n$ is convergent, then $\lim a_n = 0$.*

Proof. Observe that we can write $a_n = s_n - s_{n-1}$, where $s_n = a_0 + \cdots + a_n$ is the partial sums sequence. The fact that the series is convergent, means that s_n is a convergent sequence. Let s denote the limit of s_n. Of course, we have $\lim s_n = \lim s_{n-1} = s$ since they are both the same sequence, but shifted by 1 in the index. We therefore have $\lim a_n = \lim(s_n - s_{n-1}) = \lim s_n = \lim s_{n-1} = s - s = 0$. □

We can reformulate the previous result in a way that is more directly applicable.

Method 11.2.7 (Divergence Test)**.** *If $\lim a_n \neq 0$, then the series is divergent.*

Example 11.2.8. Let us consider the series $\sum_{n=0}^{\infty} \frac{n^2}{3n^2+1}$. Since $\lim \frac{n^2}{3n^2+1} = \frac{1}{3} \neq 0$, we have that the Divergence Test gives us that the series is divergent.

Remark 11.2.9. The Divergence Test only tells us when a series diverges, but it does not give us any information on the convergence. In fact, if $a_n \longrightarrow 0$, it does not follow that $\sum a_n$ converges as well. For instance, the harmonic series is an example of such a situation, where $\frac{1}{n} \longrightarrow 0$, but the series diverges, as we have shown directly.

Properties of convergent series

We list here three useful properties of convergent series, whose proof is simple, and left to the reader as an exercise.

Theorem 11.2.10. *Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be convergent series. Then, we have*

- *For a constant c, $\sum ca_n$ converges and $\sum ca_n = c\sum a_n$;*
- *The series $\sum(a_n+b_n)$ converges, and $\sum(a_n+b_n) = \sum a_n + \sum b_n$;*
- *The series $\sum(a_n-b_n)$ converges, and $\sum(a_n-b_n) = \sum a_n - \sum b_n$.*

11.3 Integral Test and Estimates of Sums

We now develop some techniques that allow us to determine whether a series is convergent. We begin by showing an important example.

Example 11.3.1. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

To understand whether this converges or not, let us consider the function $f(x) = \frac{1}{x^2}$. Of course, $f(n) = \frac{1}{n^2}$ whenever we pick a natural number n. So, the function in a sense generates the series. If we consider the values $n = 1, 2, \ldots$ and plot the rectangles of base $\Delta x = 1$, and heights on the function $f(x)$, we see that the series can be thought of as consisting of sums of rectangles lying below the function $f(x)$. We therefore find that the series is bounded above by the term $1 + \int_1^{\infty} \frac{1}{x^2} dx = 2$. Since the terms in the partial sums are positive, the partial sums are monotonic (increasing). From the fact that there is an upper bound given by 2, it follows from Theorem 11.1.29 that the series is convergent.

One can also proceed in a similar way for $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, but by bounding the series from below with an improper integral. The integral would now be divergent, and this would mean that the series diverges too.

The previous example is a simple version of the following important test.

Method 11.3.2 (Integral Test)**.** *Suppose that $f(x)$ is a continuous, positive and decreasing function on $[1, \infty)$, and let $a_n = f(n)$. Then, the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.*

Remark 11.3.3. The test can be applied even when the series does not start at $n = 1$, but this is what we used for the previous example. Moreover, f does not need to be decreasing everywhere. It could also be *eventually* decreasing, meaning that it is decreasing in some interval $[c, \infty)$ for some $c > 1$.

Example 11.3.4. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Of course, the terms of the series can be written by means of the function $f(x) = \frac{1}{x^2+1}$. The function f is decreasing and positive. So, we can apply the Integral Test to determine whether the series is convergent or not. We have

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1} &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1} dx \\
&= \lim_{t\to\infty} \arctan x \,|_1^t \\
&= \lim_{t\to\infty} (\arctan t - \frac{\pi}{4}) \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{4}.
\end{aligned}$$

So, the series is convergent.

Example 11.3.5. We now determine for what values of p the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

When $p < 0$, the series is obviously divergent using the Divergence Test, since $\frac{1}{n^p} \longrightarrow \infty$ in this case. Similarly, when $p = 0$ $\frac{1}{n^p} \longrightarrow 1$, and the series is divergent again.

Let us therefore consider $p > 0$. The series can be obtained by evaluating the function $f(x) = \frac{1}{x^p}$ on natural numbers, and f is continuous, positive, and decreasing on the interval $[1, \infty)$. So, we can apply the Integral Test. We know (from previous examples on improper integrals) that $\int_1^\infty \frac{1}{x^p} dx$ is convergent when $p > 1$ and it is divergent when $p \leq 1$. So, this gives us that the corresponding series is convergent when $p > 1$ and divergent when $p \leq 1$.

This series is called the *p-series*, and it is of great importance when considering the comparison tests that we will consider in the next sections.

Proof of Integral Test. The idea of the proof is the same as in the examples for $\frac{1}{n}$ and $\frac{1}{\sqrt{n}}$ already considered.

First, observe that since the series is assumed to have positive elements $a_n = f(n)$ where f is positive, it follows that the sequence of partial sums is monotonic. In fact, $s_{n+1} = a_1 + a_2 + \cdots + a_n + a_{n+1} = s_n + a_{n+1}$ and since $a_{n+1} = f(n+1) \geq 0$, $s_{n+1} \geq s_n$. So, from Theorem 11.1.29, there exists the limit of the partial sums sequence, and this is either infinity or the supremum of the set $\{s_n\}$, depending on whether $\{s_n\}$ has an upper bound or not.

Exactly as in the example we considered, we have the estimates $a_1+a_2+\cdots+a_n \leq a_1+\int_1^n f(x)dx$, because the rectangles lie all inside the area under the curve $f(x)$, and $f(x)$ is decreasing, and also $\int_1^n f(x)dx \leq a_1 + a_2 + \cdots + a_n$, because in this case the rectangles lie above the area under the curve.

Suppose that $\int_1^\infty f(x)dx < \infty$, i.e. the improper integral converges and gives a finite number. We have that

$$\sum_{k=1}^{n} a_k \leq a_1 + \int_1^n f(x)dx \leq a_1 + \int_1^\infty f(x)dx < \infty, \tag{11.8}$$

so the partial sums have an upper bound, given by $a_1 + \int_1^\infty f(x)dx$. Theorem 11.1.29 therefore guarantees that the series converges.

Conversely, assume that $\int_1^\infty f(x)dx = \infty$, i.e. the imroper integral diverges. From the estimate $\int_1^n f(x)dx \leq a_1 + a_2 + \cdots + a_n = s_n$, it follows that the partial sums s_n have no upper bound, since $\int_1^n f(x)dx \to \infty$, and $s_n \geq \int_1^n f(x)dx$. So, applying Theorem 11.1.29 we find that the series diverges to ∞. □

The Integral Test allows us to determine whether the series converges, but does not give us information on the sum of the series when this is convergent. We want to have a method that allows us to estimate the sum of a convergent series. Of course, one could sum a large number of elements of a series, say several thousands, and hope that this is a reasonable approximation. After all, the summands of a convergent series become smaller and smaller, so we have a chance that if we sum a large enough number of them, our approximation is not too bad. In order to determine whether our approximation is good, or not, we should estimate the size of the remainder, which is given by $R_n = s - s_n$, where $s = \sum_{n=0}^\infty$ is the sum of the series, and s_n is a partial sum obtained by summing all the elements up to n.

Under the same assumptions as in the Integral Test, we can find an upper bound on R_n by evaluating the integral $\int_n^\infty f(x)dx$, i.e. $R_n \leq \int_n^\infty f(x)dx$. This is because in the computation of the integral, all the rectangles (as in the Integral Test) of height $f(n)$ lie inside the area under the curve $f(x)$ for $x \geq n$. This is substantially the same argument as in the Integral Test, but where we are starting to build rectangless from an arbitrary n, instead of $n = 1$. Similarly, we can also find a lower bound as $R_n = \int_{n+1}^{\infty} f(x)dx$.

We have therefore found the following estimate.

Method 11.3.6 (Remainder Estimate for the Integral Test). *Under the same hypotheses as the Integral Test, assuming that the series converges to s, and setting $R_n = s - s_n$, we have the estimates*

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx. \tag{11.9}$$

This result allows us to approximate a sum within a specified error. In fact, by taking n large enough, we can control how small R_n becomes, and this gives us the accuracy of the approximation $\sum_{n=0}^{\infty} a_n \approx \sum_{k=0}^{n} a_k$, obtained by summing elements of the series only up to n, rather than up to ∞.

11.4 Comparison Tests

In this section we will discuss more criteria to determine that a series is convergent. Here we consider only series with non-negative terms. This means that the partial sums are monotonic sequences, and by Theorem 11.1.29 the series are either convergent, or divergent to ∞, and this depends on whether the series has a supremum or not.

Direct Comparison

The first result is very natural. If we have two series, one of which is always smaller than the other, then the convergence of the larger ones implies convergence of the smaller one. This is because there will be an upper bound on the smaller series, and therefore it will have to converge (see discussion on the use of Theorem 11.1.29 above). Also, if the smaller one diverges, the larger one has to diverge as well following a similar reasoning. We have the following result.

Method 11.4.1 (Direct Comparison Test). *Suppose that $\sum a_n$ and $\sum b_n$ are series with non-negative terms, such that $a_n \leq b_n$ for all n (or eventually). Then,*

(i) If $\sum b_n$ is convergent, $\sum a_n$ is convergent as well.

(ii) If $\sum a_n$ is divergent, $\sum b_n$ is divergent as well.

Proof. As already pointed out, both $\sum a_n$ and $\sum b_n$ must have a limit, which is either finite or infinite, by Theorem 11.1.29, because their partial sums are monotonic increasing. Whether the limit is finite or infinite depends only on them having a finite supremum or not. We consider the case where $a_n \leq b_n$ for all n, and not for n starting at some point (i.e. eventually). The latter case is substantially the same.

Let us prove (i). Since $\sum b_n$ converges, it has a finite supremum and therefore an upper bound $M > 0$. Since $a_n \leq b_n$ for all n, it follows that $\sum_{n=1}^{k} a_n \leq \sum_{n=1}^{k} b_n \leq M$ for all n, which means that the partial sums of a_n have an upper bound and therefore they have a finite supremum. This means that $\sum a_n$ is convergent by Theorem 11.1.29.

Let us prove (ii). Since $\sum a_n$ is divergent, its partial sums have no upper bound. So, from $\sum_{n=1}^{k} a_n \leq \sum_{n=1}^{k} b_n$ it follows that the partial sums of b_n have no upper bound as well (or this would be an upper bound for the partial sums of a_n which we know do not have one). It follows that the partial sums of b_n are not bounded above, and therefore the sequence of partial sums is divergent to ∞ by Theorem 11.1.29. □

The Direct Comparison Test is useful because we can compare series with some series whose convergence and divergence we already estabilished. For instance, typical series to use for comparison are the p-series $\sum \frac{1}{n^p}$, and the geometric series $\sum ar^n$.

Example 11.4.2. Consider the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$. Of course, the integral test can be applied in this case. However, there is a very simple way of using the Direct Comparison Test in this case.

In fact, observe that $\ln n > 1$ for all $n \geq 3$. So, we have the inequality $\frac{\ln n}{n} > \frac{1}{n}$ for all $n \geq 3$. Since we have already shown that $\sum \frac{1}{n}$ is divergent, it follows that $\sum \frac{\ln n}{n}$ is divergent as well.

11.4.1 Limit Comparison Test

The limit comparison test allows us to compare the convergence and divergence of series by computing a limit of their quotients. This is more useful than the Direct Comparison Test when it is less obvious that a certain inequality between the terms of the series exists.

Method 11.4.3 (Limit Comparison Test)**.** *Let $\sum a_n$ and $\sum b_n$ be series with non-negative terms, where $b_n \neq 0$ for all n (or eventually for n large enough). Then, if*

$$\lim \frac{a_n}{b_n} = c,$$

with $c > 0$ a finite number, then both series have the same behavior, i.e. they are either both convergent or both divergent to ∞.

Proof. Since $0 < c < \infty$, we can find positive numbers m and M such that $m < c < M$. Since $\frac{a_n}{b_n} \longrightarrow c$, we have that we can find a ν such that $m < \frac{a_n}{b_n} < M$ for $n \geq \nu$, from the definition of limit. Therefore, we have that eventually (i.e. for $n \geq \nu$) $mb_n < a_n < Mb_n$. So, if $\sum b_n$ is convergent, it follows that also $\sum Mb_n$ is convergent by Theorem 11.2.10. By the Direct Comparison Test, then we will have that $\sum a_n$ is convergent. If b_n diverges, so does $\sum mb_n$ (again by Theorem 11.2.10, since otherwise $\sum b_n = \sum \frac{1}{m} mb_n$ would be convergent) and using the Direct Comparison Test we find that $\sum a_n$ diverges as well. This completes the proof. □

Example 11.4.4. Consider the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$.

The behavior of the fraction $\frac{2n^2+3n}{\sqrt{5+n^5}}$ is dominated by the largest terms in the numerator and the denominator, which are $2n^2$ and $\sqrt{n^5}$, respectively. So, we compare the fraction with the series $\frac{2n^2}{\sqrt{n^5}}$. Also, observe that $\sum \frac{2n^2}{\sqrt{n^5}} = \sum 2\frac{1}{\sqrt{n}}$ is a divergent series, since we know that p-series with $p \leq 1$ diverge.

We compare $\frac{2n^2+3n}{\sqrt{5+n^5}}$ and $2\frac{1}{\sqrt{n}}$. By taking the quotient we find

$$\begin{aligned}
\frac{\frac{2n^2+3n}{\sqrt{5+n^5}}}{2\frac{1}{\sqrt{n}}} &= \frac{2n^2+3n}{\sqrt{5+n^5}} \cdot \frac{\sqrt{n}}{2} \\
&= \frac{n^{\frac{5}{2}} + \frac{3}{2}n^{\frac{3}{2}}}{\sqrt{5+n^5}} \\
&= \frac{1+\frac{3}{2n}}{\sqrt{\frac{5}{n^5}+1}} \longrightarrow \frac{1+0}{\sqrt{0+1}} = 1 > 0.
\end{aligned}$$

So, the Limit Comparison Test allows us to say that $\sum \frac{2n^2+3n}{\sqrt{5+n^5}}$ and $\sum 2\frac{1}{\sqrt{n}}$ are either both convergent or both divergent. Since $\sum 2\frac{1}{\sqrt{n}}$ is divergent, as already pointed out, it means that the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ is divergent as well.

11.5 Alternating Series and Absolute Convergence

The convergence tests we have discussed in the previous section refer to series that have positive terms. Of course, one might wonder how to deal with the case of series that also have negative terms too. If the series has only negative terms, then one can consider the series given by multiplying all terms by a negative sign. This reduced the problem to what we have done for positive terms, since if the series $\sum(-a_n)$ converges, then the series $\sum a_n$ converges as well. If a series has only finitely many negative terms, then one can simply proceed as in the previous section, since adding finitely many negative terms would not change conveergence. The real issue is when the series has infinitely many negative terms and positive terms. A typical example is a series with alternating positive and negative terms, for example $\sum(-1)^n\frac{1}{n}$. For such series, we have the following test.

Method 11.5.1 (Alternating Series Test)**.** *If the alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$, with all $a_n > 0$, satisfies the conditions*

(i) $a_{n+1} \leq a_n$,

(ii) $\lim a_n = 0$,

then the series is convergent.

Proof. We consider first the odd partial sums s_{2n+1}, obtained by adding up to an odd index of the series, e.g. s_1, s_3, s_5, s_7 and so on. We have $s_1 = a_0 - a_1$, $s_e = a_0 - a_1 + a_2 - a_3 = s_1 + (a_2 - a_3)$, and similarly $s_{2n+1} = s_{2n-1} + (a_{2n} - a_{2n+1}) \geq s_{2n-1}$. This shows that the sequence of partial sums (with odd number of summands) is increasing. However, the grouping of the terms can be also written as $s_{2n+1} = a_0 - (a_1 - a_2) - (a_3 - a_4) - \cdots - (a_{2n-1} - a_{2n}) - a_{2n+1}$. Since $a_{n+1} \leq a_n$, each term $a_1 - a_2$, $a_3 - a_4$ and so on is negative. Therefore, the second way of writing s_{2n+1} shows that we have $s_{2n+1} \leq a_0$. This means that s_{2n+1} is a monotonic increasing sequence that has an upper bound. It must therefore be convergent by Theorem 11.1.29. We denote $\lim s_{2n+1} = s$ the limit of the sequence of partial sums. We now consider the even terms of the partial sums s_{2n}. In this case we can write $s_{2n} = s_{2n-1} + a_{2n}$ which is a sum of an odd partial sum and an extra term.

Taking limits of both sides of the previous equation, and considering that $\lim a_{2n} = \lim a_n = 0$ by assumption, we find that $s_{2n} \longrightarrow s$ as well. This means that both odd and even terms of the partial sums converge to s. It can be shown (and it is left as a simple exercise to the reader) that this means that $\lim s_n = s$, therefore completing the proof. □

As usual, we can have the series starting at n different from 0, without changing the previous result.

Example 11.5.2. Let us consider the sequence $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$. We already know that the series $\sum \frac{1}{n}$ is divergent. However, $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ satisfies the conditions of the Alternating Series Test. In fact, obviously one has $\frac{1}{n+1} \leq \frac{1}{n}$, and also $\frac{1}{n} \longrightarrow 0$. So, the test gives us that the series is convergent.

The following definition is of great importance.

Definition 11.5.3. The series $\sum a_n$ is said to be *absolutely convergent* if the series of absolute values $\sum |a_n|$ is convergent. A series is *conditionally convergent* if $\sum a_n$ is convergent, but $\sum |a_n|$ is not convergent.

Example 11.5.4. From the previous example, we have seen that $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ is convergent. However, $\sum_{n=1}^{\infty} |(-1)^n \frac{1}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent, as we have previously seen. So, $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ is an example of a conditionally convergent series.

On the contrary, $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$ is absolutely convergent, since $\sum_{n=1}^{\infty} |(-1)^n \frac{1}{n^2}| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p=series with $p = 2 > 1$.

Theorem 11.5.5. *If a series is absolutely convergent, then it is convergent.*

Proof. We observe that $a_n \leq |a_n|$ is always true, since if a_n is positive, then $|a_n| = a_n$, and if a_n is negative, then $a_n < 0 < |a_n|$ holds true. Moreover, $a_n + |a_n| \geq 0$ since if a_n is negative then $|a_n| = -a_n$ and we have $a_n + |a_n| = 0$, while if $a_n \geq 0$ we just have $a_n + |a_n| = a_n + a_n \geq 0$. Therefore, $0 \leq a_n + |a_n| \leq 2|a_n|$. In the assumption that $\sum a_n$ is absolutely convergent, we obtain that also $\sum 2|a_n|$ is convergent (since it is the series of absolute values just multiplied by a number, see Theorem 11.2.10). By the Direct Comparison Test, it follows that $\sum(a_n + |a_n|)$ is convergent. This means that $\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$ is the difference of two convergent series, which means it is convergent (again, see Theorem 11.2.10). This completes the proof. □

Example 11.5.6. We want to show that the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent.

We cannot use the Alternating Series Test, since the series has negative and positive terms, but they do not alternate. Let us consider the series of absolute values $\sum_{n=1}^{\infty} |\frac{\cos n}{n^2}|$. Since $|\cos x| \leq 1$ always (so in particular for all n), we can write $|\frac{\cos n}{n^2}| \leq \frac{1}{n^2}$. We know that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. So, from the Direct Comparison Test, it follows that $\sum_{n=1}^{\infty} |\frac{\cos n}{n^2}|$ is also convergent, since all the terms are smaller than the terms of $\sum \frac{1}{n^2}$ which is convergent. It follows that $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent. Then, by Theorem 11.5.5, it follows that it is also convergent.

11.6 The Ratio and Root Tests

One problem with the tests discussed until now, execpt the Alternating Series Test, is that they are based on comparisons. This means, that we need to compare our series with some other series whose behavior is known to us. In this section we will see that there are some techniques to determine convergence/divergence that do not depend on comparisons with other series, but only refer to the series whose behavior is the object of our study.

The Ratio Test

Method 11.6.1 (Ratio Test)**.** *The following facts hold.*

(i) If $\lim|\frac{a_{n+1}}{a_n}| = L < 1$*, then the series* $\sum a_n$ *is absolutely convergent, and therefore convergent.*

(ii) If $\lim|\frac{a_{n+1}}{a_n} = L > 1$*, where* L *can also be* ∞ *here, then the series* $\sum a_n$ *is not convergent. If* a_n *is positive for all* n *(or eventually), then the series is divergent to* ∞*.*

Proof. We first prove (i). From the fact that $L < 1$, it follows that we can find a number r such that $L < r < 1$ holds. The idea is to compare the given series with a geometric series with this r, which is convergent because $r < 1$. From the fact that $\lim|\frac{a_{n+1}}{a_n}| = L < r$, it follows that for n large enough, say $n \geq \nu$, we will have $|\frac{a_{n+1}}{a_n}| < r$. Therefore, we have $|a_{n+1}| < a_n r$. Then, taking $n = \nu$ we have

$$\begin{aligned}
|a_{\nu+1}| &< |a_\nu|r \\
|a_{\nu+2}| &< |a_{\nu+1}|r < |a_\nu|r^2 \\
|a_{\nu+3}| &< |a_{\nu+2}|r < |a_\nu|r^3 \\
&\vdots \\
|a_{\nu+k}| &< |a_\nu|r^k
\end{aligned}$$

Observe that the series $\sum_{k=1}^{\infty}|a_\nu|r^k$ is a geometric series which is convergent since $r < 1$. Therefore, we have that the series $\sum_{n=\nu+1}^{\infty}|a_n| = |a_{\nu+1}+\cdots+a_{\nu+k}+\cdots$ is also convergent because of the Direct Comparison Test, since each of its terms $a_{\nu+k}$ are smaller than $|a_\nu|r^k$. Since the series $\sum_{n=\nu+1}^{\infty}|a_n|$ differs from $\sum_{n=1}^{\infty}|a_n|$ only by finitely many terms, ν of them, if follows that $\sum_{n=1}^{\infty}|a_n|$ is convergent too, since finitely many terms cannot change the character of a series. So, $\sum_{n=1}^{\infty}a_n$ is absolutely convergent, and therefore convergent by Theorem 11.5.5.

We now prove (ii). From the fact that $L > 1$, we find that eventually $|a_{n+1} > a_n$ for all $n \geq \nu$. Therefore, the limit $\lim a_n$ cannot be zero, and the Divergence Test implies that the series cannot be convergent. If the terms are all non-negative, the sequence of partial sums has a limit that cannot be finite, and it is therefore infinite by Theorem 11.1.29. This completes the proof. □

Remark 11.6.2. Observe that when we are unlucky enough to get $\lim|\frac{a_{n+1}}{a_n}| = 1$, the Ratio Test is inconclusive, meaning that we cannot determine the behavior of the series using it (other methods will be needed).

Example 11.6.3. Consider the series $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$. We want to determine whether it is convergent or divergent.

We have

$$\begin{aligned}
|\frac{a_{n+1}}{a_n}| &= |\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{(-1)^n\frac{n^3}{3^n}}| \\
&= \frac{(n+1)^3}{3^{n+1}}\frac{3^n}{n^3} \\
&= \frac{1}{3}(\frac{n+1}{n})^3 \\
&= \frac{1}{3}(1+\frac{1}{n})^3 \longrightarrow \frac{1}{3} < 1.
\end{aligned}$$

The Ratio Tesst therefore shows that the series is absolutely convergent, and therefore it is convergent.

The proof of the following test is completely analogous to the proof of the Ratio Test, and it is left as an exercise to the reader.

Method 11.6.4 (Root Test)**.** *The following facts hold.*

(i) If $\lim \sqrt[n]{|a_n|} = L < 1$*, then the series* $\sum a_n$ *is absolutely convergent, and therefore convergent.*

(ii) If $\lim \sqrt[n]{|a_n|} = L > 1$*, where* L *can also be* ∞ *here, then the series* $\sum a_n$ *is not convergent. If* a_n *is positive for all* n *(or eventually), then the series is divergent to* ∞*.*

Remark 11.6.5. As in the case of the Ratio Test, when the limit is 1, the Root Test will not give any conclusive result, and other methods need to be used.

11.7 Power Series

We now study power series, which are series where the variable x appears as well. In a sense, one can think of a power series as an object that gives a series whenever we choose a certain value for x. Of course, understanding for what values of x this series is convergent is of great importance. For all those x such that the series is convergent, we obtain a number, and this means that we can identify this with a function – this object takes an input, a numerical value of x, and returns a number, the number of convergence of the series.

We begin with a definition.

Definition 11.7.1. A *power series* is a series of type $\sum_{n=0}^{\infty} c_n x^n$, where x is a variable and the constants c_n are called *coefficients* of the series. More generally, a power series has the form $\sum_{n=0}^{\infty} c_n(x-a)^n$, and in this case we say that the power series is centered at a.

As mentioned above, for all x such that $\sum_{n=0}^{\infty} c_n x^n$ is convergent, we obtain a function by setting $f(x) = \sum_{n=0}^{\infty} c_n x^n$. Such a function is very similar to a polynomial, with the fundamental difference that the summands are infinitely many.

Example 11.7.2. Taking $c_n = 1$ for all n, we obtain the power series $\sum_{n=0}^{\infty} x^n$ which is a geometric series. We already know that it is convergent whenever $-1 < x < 1$. So, we have a function $f(x) = \sum_{n=0}^{\infty} x^n$ which is well defined on the open interval $(0,1)$. Since we also know what numbers the geometric series converges to, we can even give in this case an analytical expression of the corresponding function. We have $f(x) = \frac{1}{1-x}$. However, this is not always possible, since it is not simple to determine where a series converges to.

Example 11.7.3. We want to find for what values of x the power series $\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$ converges. Here, $c_n = \frac{1}{n}$, and $a = 3$.

Let us use the Ratio Test. Given a chosen x, the general term of the series can be written as $a_n = \frac{1}{n}(x-3)^n$. So, we have

$$\begin{aligned} |\frac{a_{n+1}}{a_n}| &= |\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}| \\ &= |\frac{1}{1+\frac{1}{n}}(x-3)| \longrightarrow |x-3|. \end{aligned}$$

The Ratio Test tells us that whenever $|x-3| < 1$, the series is absolutely convergent, and therefore convergent. Also, when $|x-3| > 1$ the series is divergent. Since $|x-3| < 1$ whenever $2 < x < 4$, we have found that the series is convergent when x is in $(2,4)$, and divergent whenever x is in $(-\infty, 2) \cup (4, \infty)$. However, $x = 2$ and $x = 4$ corresopnds to cases when $|x-3| = 1$, and we know that the Ratio Test does not give any conclusions in this case. Therefore, we have to analyze the cases $x = 2$ and $x = 4$ separately. When $x = 4$, one can immediately see that the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$ which we already know is divergent (this is a p-series). When $x = 2$, the series becomes $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$. We have previously seen, by means of the Alternating Series Test, that this series is convergent. This completes all the possible cases for x. Therefore, the series is convergent for x in $[2,4)$, and divergent otherwise.

The faact that in the previous cases the set of points of x such that the power series is convergent is an interval is not an accident. In fact, we have the following result.

Theorem 11.7.4. *Let $\sum_{n=0}^{\infty} a_n(x-a)^n$ be a power series. Then there are only three possibilities:*

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There exists a number $\rho > 0$ such that the series converges when $|x-a| < \rho$, and diverges when $|x-a| > \rho$.

In fact, one can unify the last two cases in the previous result, adding tan extreme case. Taking $\rho = \infty$, one has that $|x-a| < \infty$ always, so the series is convergent for all x. In case (i) one also says that $\rho = 0$ (even though $|x-a| < 0$ is not true). The number (or infinity) ρ is called the radius of convergence of the series, and the corresponding interval is called the interval of convergence. So, Theorem 11.7.4 states that given a power series, the values of x such that the series converges lie in an interval (the interval of convergence). In case (i), this interval has zero radius, and it consists of a single point $x = a$. In case (ii), the series converges over the whole $(-\infty, \infty)$ and the radius of convergence is $\rho = \infty$. In case (iii), the radius is a positive (finite) number, and the interval of convergence is $(a-\rho, a+\rho)$. The boundary cases $x = a \pm \rho$ need to be determined on a case by case basis, as any type of behavior might happen.

Example 11.7.5. We want to find the radius and interval of convergence of the series $\sum_{n=0}^{\infty}(-3)^n \frac{x^n}{\sqrt{n+1}}$.

We use again the Ratio Test, where $a_n = (-3)^n \frac{x^n}{\sqrt{n+1}}$. We have

$$\begin{aligned}
|\frac{a_{n+1}}{a_n}| &= |\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}| \\
&= |-3x\sqrt{\frac{n+1}{n+2}}| \\
&= 3|x|\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} \longrightarrow 3|x|.
\end{aligned}$$

Therefore, by the Ratio Test, whenever $3|x| < 1$ the series converges. This means that the series converges when $-\frac{1}{3} < x < \frac{1}{3}$. Also, the series diverges when $x < -\frac{1}{3}$ and $x > \frac{1}{3}$. Therefore, the radius of convergence is $\rho = \frac{1}{3}$. The only two undetermined cases are $x = -\frac{1}{3}, \frac{1}{3}$. As usual, they need to be handled separately. When $x = -\frac{1}{3}$ the series becomes $\sum \frac{1}{\sqrt{n+1}}$ which is divergent (this is a p-series with $p = \frac{1}{2} < 1$). When $x = \frac{1}{3}$ the series becomes $\sum \frac{(-1)^n}{\sqrt{n+1}}$ which is convergent by the Alternating Series Test. We have therefore found that the interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$.

11.8 Power Series Expansions

The scope of this section is to investigate properties of functions that are written as power series. For example, we have already seen that in the interval $(-1, 1)$, the function $f(x) = \frac{1}{1-x}$ can be written as a power series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$.

In such a situation, when a function $f(x)$ can be written as a power series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ centered at some a, we say that $f(x)$ has a power series expansion, or a power series representation.

Example 11.8.1. Let us consider the function $f(x) = \frac{1}{1+x^2}$. We want to express the function as a power series in the interval of convergence of the series.

Observe that we can write $f(x) = \frac{1}{1-(-x)^2}$. So, we can use the previous result $\frac{1}{1-x}$ where we are now replacing x by $(-x)^2$. We obtain $f(x) = \frac{1}{1-(-x)^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$. The interval of convergence of the power series, which is also the interval where $f(x)$ can expressed as a power series is the set of points where $|-x^2| < 1$, which gives us $|x| < 1$, so $(-1, 1)$ again.

One property of polynomials is that differentiating and integrating them is very easy. This fact is true even when considering power series. Observe that this is not automatically true, since they are infinite sums, and therefore is something that does not follow from the properties of polynomials. We have the following result.

Theorem 11.8.2. *Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ be a function expanded as a power series with radius of convergence ρ. Then $f(x)$ is differentiable in $(a-\rho, a+\rho)$ and*

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

(ii) $\int f(x)dx = c + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = c + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$.

Moreover, the radii of convergence of the two power series in (i) and (ii) are both ρ.

Remark 11.8.3. Observe that while the radii of convergence are the same, the endpoints of the intervals might be points where convergence changes. So, the original series might converge at the point $a+\rho$, but the differentiated series does not converge there.

Example 11.8.4. We want to find the power series representation, if it exists, of the function $f(x) = \arctan(x)$.

Observe that $f'(x) = \frac{1}{1+x^2}$. Therefore, we can expand $f'(x)$ using the results for the previous example. We have $f'(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n}$. Applying Theorem 11.8.2 we can now integrate the series to obtain the expansion of $f(x)$. We get

$$\begin{aligned}\arctan x &= \int \frac{1}{1+x}dx \\ &= \int \sum_{n=0}^{\infty}((-1)^n x^{2n})dx \\ &= c + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.\end{aligned}$$

The radius of convergence of $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ is 1, as we previously determined, and therefore the radius of convergence of $c + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ is 1 as well by Theorem 11.8.2.

11.9 Taylor and Maclaurin Series

In this section we discuss how to find power series expansions of functions.

We start with the assumption that a function $f(x)$ can be written as a power series in an interval of radius ρ around $x=a$. So, for all x such that $|x-a|<\rho$, it holds

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

How can we determine the coefficients c_n in a more systematic way than we did in the previous section?

Theorem 11.9.1. *Assume that f has a power expansion at $x=a$, so that*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

for $|x-a|<\rho$. Then, the coefficients c_n are determined by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}. \tag{11.10}$$

Proof. Observe that upon evaluating f in $x=a$ (the center of the interval), all terms in the power series containing $(x-a)^n$ will vanish. In other words, it must hold $f(a)=c_0$. So, just by plugging $x=a$ we have determined the first coefficient. Of course, our luck with this approach terminates here, it seems. However, we can apply Theorem 11.8.2 to turn c_1 into the lowest degree term by taking a derivative! In this way, we will be able to proceed as before and get c_1. In detail, differentiating $f(x)$ using Theorem 11.8.2 we get

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots,$$

which shows that all terms but c_1 are multiplied by some power of $(x-a)$. This means that $f'(a) = c_1$, since $(x-a)$ vanishes when $x = a$.

How about c_2? The idea is simple. We play again the same game: First we differentiate, so that c_2 has lowest degree (i.e. 0) in $(x-a)$, and then we evaluate at $x = a$. Using again Theorem 11.8.2 we get

$$f''(x) = 2c_2 + 2\cdot 3\cdot c_3(x-a) + 3\cdot 4\cdot c_4(x-a)^2 + 4\cdot 5\cdot c_5(x-a)^3 + \cdots$$

from which, upon evaluating at $x = a$ we obtain $f''(a) = 2c_2$, and therefore $c_2 = \frac{f''(a)}{2}$.

In general, to have the term c_n as the degree zero one, i.e. the one which is not multiplied by any powers of $(x-a)$, we need to take n derivatives of $f(x)$. Each time we differentiate, using the power rule (for infinite series, i.e. Theorem 11.8.2) we will have all the numbers from 2 to n multiplied together: $2\cdot 3\cdot 4\cdots(n-1)\cdot n$. This number is the factorial $n!$. So, to obtain c_n we just need to evaluate the n^{th} derivative of f at $x = a$, and divide by $n!$ and we get

$$c_n = \frac{f^{(n)}(a)}{n!},$$

which is precisely Equation (11.10). □

Definition 11.9.2. We have therefore found that if a function can be expanded in a power series about $x = a$, it has the form

$$f(x) = \sum_{n=0}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{11.11}$$

This is called the Taylor series (or Taylor expansion) of f about a (or centered at a, or at a). In the particular case where $a = 0$, the series is called Maclaurin series, and it takes the simpler form

$$f(x) = \sum_{n=0}^{n} \frac{f^{(n)}(0)}{n!}x^n. \tag{11.12}$$

Example 11.9.3. Let $f(x) = e^x$ be the exponential function.

We want to find the Maclaurin series for the exponential. We know that taking derivatives of the exponential, gives again the exponential. So, we know that $f^{(n)}(x) = e^x$ for all n. Therefore, $f^{(n)}(0) = 1$ for all n. Using Equation (11.12) with $f^{(n)}(0) = 1$ we get the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{1}{n!}x^n.$$

Let us now consider the convergence of this series by means of the Ratio Test. We have

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| \\ &= \frac{|x|}{n+1} \longrightarrow 0 < 1. \end{aligned}$$

This limit is independent of the value of x. In other words, by the Ratio Test, no matter what x we pick, the series will be convergent. So, $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ defines a function over the whole real numbers.

At this point, if we only knew that the exponential function has a Maclaurin expansion, that would mean that what we have found is exactly its expansion (observe that Theorem 11.9.1 states that if f has an expansion, then it is given by the formula we have found, but we do not yet know whether e^x has an expansion).

From the previous example, it has become clear that our question has now become: When can a function be written as a power series. Of course, from Theorem 11.8.2 such a function needs to have infinitely many derivatives, since each time we have a series, we can take its derivative, and we can keep doing this over and over.

Our objective is to find when

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Since this is a series, equality means that the limit of the partial sums converges to $f(x)$. Let us call the partial sums

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k && (11.13)\\ &= f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. && (11.14) \end{aligned}$$

So, $T_n(x)$ is a polynomial of degree n, called the n^{th}*-degree Taylor polynomial of* f *at* a. The whole Taylor series can be written by a Taylor polynomial up to n, plus the remainder, i.e. all the higher powers that are not included in $T_n(x)$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = T_n(x) + \sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

We therefore set $R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(x) - T_n(x)$, which is the remainder of the Taylor series. In other words, $R_n(x)$ is what is left of the function once we take out the Taylor polynomial of degree n. In the assumption that $\lim_n R_n(x) = 0$ for all x in some interval, then we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n &= \lim_n T_n(x) \\ &= \lim_n [f(x) - R_n(x)] \\ &= f(x) - \lim_n R_n(x) \\ &= f(x). \end{aligned}$$

So, if the remainder goes to zero for all x, the Taylor series converges to $f(x)$ for all x, which is what we wanted to understand in the first place. We therefore have the following theorem.

Theorem 11.9.4. *If* $\lim_n R_n(x) = 0$ *for* $|x-a| < \rho$, *then* f *is equal to its Taylor series on the interval* $(a-\rho, a+\rho)$.

We now have naturally found another problem: When does $\lim_n R_n(x) = 0$ hold? The following result is a powerful tool to answer this question based on a given function f.

Theorem 11.9.5 (Taylor's Inequality)**.** *If $|f^{(n+1)}(x)| \leq M$ for all x such that $|x - a| \leq f$, for some $d > 0$, then*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1}$$

for $|x - a| \leq d$.

Proof. We prove the result for $n = 1$, since the same procedure (performing more integrations!) can be performed when $n > 1$. So, we assume that $|f''(x)| \leq M$ for $|x - a| \leq d$, and therefore $f''(x) \leq M$. Let us consider $a \leq x \leq a + d$. In fact, an analogous proof works when $a - d \leq x \leq a$. Integrating $|f''(x)| \leq M$ for $|x - a| \leq d$ on both sides, we get

$$\int_a^x f''(t)dt \leq \int_a^x M dt.$$

Since $f'(x)$ is an antiderivative of $f''(x)$, we have

$$f'(x) - f'(a) \leq M(x - a),$$

from which

$$f'(x) \leq f'(a) + M(x - a).$$

Integrating again the inequality we have

$$\int_a^x f'(t)dt \leq \int_a^x [f'(a) + M(t - a)]dt,$$

which gives us

$$f(x) - f(a) \leq f'(a)(x - a) + M\frac{(x - a)^2}{2},$$

and therefore

$$f(x) - f(a) - f'(a)(x - a) \leq M\frac{(x - a)^2}{2}. \tag{11.15}$$

Observe now that $T_1(x) = f(a)+f'(a)(x-a)$, and therefore $f(x)-f(a)-f'(a)(x-a) = f(x)-T_1(x)$. But also, we have seen before the proof of Theorem 11.9.4 that $f(x) - T_n(x) = R_n(x)$. So, using (11.15) we find that

$$R_1(x) \leq \frac{M}{2}(x - a)^2.$$

We can repeat the same procedure for $f''(x) \geq -M$, which also follows from $|f''(a)| \leq M$, and we would obtain

$$R_1(x) \geq -\frac{M}{2}(x - a)^2.$$

So, combining the previous two inequalities we have found

$$|R_1(x)| \leq |\frac{M}{2}(x - a)^2|,$$

which completes the proof.

Observe that here the crucial thing is that all the derivatives are bounded by some constant, so that the number $M = e^d$ does not depend on n. □

Remark 11.9.6. Theorem 11.9.5 is useful because of the limit $\lim_n \frac{x^n}{n!} = 0$. So, the right hand side of the inequality of the theorem goes to zero as n goes to infinity.

Example 11.9.7. Let us go back to the example of the exponential function e^x, and let us verify that this is the limit of its Maclaurin series.

When $f(x) = e^x$, all derivatives of f give also e^x: $f^{(n)}(x) = e^x$. So, we can take M to be e^d, and apply Theorem 11.9.5 (with $a = 0$) to get

$$|R_n(x)| \leq \frac{e^d}{(n+1)!}|x|^{n+1}$$

whenever $|x| \leq d$. Since $\lim_n \frac{r^d}{(n+1)!}|x|^{n+1} = 0$, it follows that $\lim_n |R_n(x)| = 0$ by the Squeeze Theorem, and therefore also $\lim_n R_n(x) = 0$. Since for any x we can find some $d > x$, we can proceed as above, we find that $\lim_n |R_n(x)| = 0$ for all x. Applying Theorem 11.9.4 it follows that the exponential function equals its Maclaurin series for all x, i.e. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

We now consider some other examples of Taylor series of particularly important functions, such as sine and cosine.

Example 11.9.8. Let us consider the sine function. We compute the Maclaurin series of $f(x) = \sin(x)$. We have

$$\begin{aligned} f(x) &= f(0) + \frac{f'(0)}{1!} + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \frac{f^{(iv)}(0)}{4!} + \frac{f^{(v)}(0)}{5!} \cdots \\ &= \sin(0) + \frac{\cos(0)}{1!} + \frac{-\sin(0)}{2!} + \frac{-\cos(0)}{3!} + \frac{\sin(0)}{4!} + \frac{\cos(0)}{5!} + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \end{aligned}$$

where in the last step we have used the fact that the derivatives of sine and cosine repeat the same pattern over and over.

Since the derivatives of sine are all bounded (they are all sines and cosines with a ± sign, and therefore their absolute value is bounded by 1), we can repeat the same reasoning that we applied for the exponential function, finding that the Taylor series of sine converges to $\sin(x)$ for all x. In other words, we have

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

for all x.

A similar approach can be used for the cosine function, giving

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

for all x.

Chapter 12

Parametric Equations and Polar Coordinates

Curves in space, such as a circle or an ellipse, or more generally curves that are not geometric objects in space, do not usually correspond to a function relating x and y. In other words, you cannot write $y = f(x)$. However, it turns out that it is still possible to treat them using calculus. Consider for instance the motion of a particle p in a 2 dimensional plane. Generally speaking, if the particle has an arbitrary trajectory, we cannot treat this as a function. However, both coordinates, x and y are expressed in terms of the time coordinate t. So, x and y are functions of t. In other words, for such a curve, while the trajectory is not a function, the coordinates are indeed functions of t, which is the independent variable. This is a curve defined by a parametric equation.

12.1 Curves and Parametric Equations

We suppose that x and y are both functions of a variable t, which is called the parameter. Then, we have $x = f(t)$ and $y = g(t)$. These are called parametric equations. When we select a value for t, we obtain a corresponding point (x, y) in the plane as $(x, y) = (f(t), g(t))$. The latter is called a parametric curve. While t is very often time, this is not always the case, so it is useful to imagine trajectories of objects paramtrized by time, but this is not the most general situation. For instance, t might very well be an angle, in which case people usually write θ for it.

Example 12.1.1. A very simple example of parametric curve is given by the circle. Here $(x, y) = (\cos\theta, \sin\theta)$.

Example 12.1.2. Another example is the Lissajous figure, defined as $(x, y) = (\cos\theta, \sin 2\theta)$.

We now consider the notion of tangent lines to parametric curves. The idea here lies in the fact that we can locally write the cruve as a function y of x, even though this is no true for the whole curve. Locally here means that given a point, we can do this around this point. For instance, imagine to take the circle. We know that the circle can be written as a union of two functions $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$, which are obtained from the equation $x^2 + y^2 = 1$ by solving for y. In a parametric curve, we can do this locally, but x will still depend on t. When we write

$y = q(x(t))$, for a function q, this means that we can differentiate y with respect to t by using the chain rule. We have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

Now, in the assumption that $\frac{dx}{dt} \neq 0$, we find

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \tag{12.1}$$

This equation allows us to find the slope of the tangent to the curve at a point. For the second derivative, one proceed in exactly the same way, but replacing y by $\frac{dy}{dx}$. We get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}. \tag{12.2}$$

Example 12.1.3. We want to study the curve C: $x = t^2$, $y = t^3 - 3t$.

Observe first that when $t = \pm\sqrt{3}$, the value of y is zero. So, the point $(3, 0)$ is repeated twice.

We compute the slope of C using Equation (12.1). We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}.$$

We see that $\frac{dy}{dx} = 0$ when $t = \pm 1$, while the derivative is not defined when $t = 0$. At the point $(3, 0)$ the curve C has two tangents, since the curve passes through the point twice. The derivatives have values $\frac{dy}{dx} = \pm\sqrt{3}$ for $t = \pm\sqrt{3}$. We can also study the concavity of C by considering the second derivative through Equation (12.2). We have

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dy}{dt}} = \frac{\frac{d}{dt}(\frac{3t^2-3}{2t})}{2t} = \frac{3t^2 + 3}{4t^3}.$$

It follows that the concavity is downward when $t < 0$ and upward when $t > 0$.

Exercise 12.1.4. Draw the curve of the previous example.

To find the area under the curve $x = f(t)$ and $y = g(t)$, we can use the integration

$$A = \int_a^b y dx = \int_\alpha^\beta g(t)f'(t)dt,$$

where α is the value of t such that $x = a$, β is the value of t such that $x = b$, and we have used the fact that $x = f(t)$ means that $dx = f'(t)dt$. To find the area enclosed in a curve, we can take a difference of areas computed using the previous formula.

We can also compute the arc length of a curve C given in parametric form $x = f(t)$, $y = g(t)$. In fact, we can write our curve locally as a union of curves $C_1, \dots, C_k$ such that in each curve C_i is given by a function $y = F(x)$. Then, we can use the formula for the computation of the arc length where $x = f(t)$. Then, assuming that $a_i = f(\alpha_i)$ and $b_i = f(\beta_i)$, we have

$$\begin{aligned} L_i &= \int_{a_i}^{b_i} \sqrt{1 + (\frac{dy}{dx})^2} dx \\ &= \int_{\alpha_i}^{\beta_i} \sqrt{1 + (\frac{\frac{dy}{dt}}{\frac{dx}{dt}})^2 \frac{dx}{dt}} dt \\ &= \int_{\alpha_i}^{\beta_i} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt. \end{aligned}$$

Once we put together all the terms L_i we obtain the following result.

Theorem 12.1.5. *If a curve C is given by parametric equations $x = f(t)$ and $y = g(t)$ where f and g have continuous derivative, and where $\alpha \leq t \leq \beta$, and C is traversed exactly once as t increases from α to β, then the length of C is*

$$L = \int_{\alpha}^{\beta} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt.$$

The polar coordinate system is based on the observation that given a point $P \equiv (x, y)$ in the Cartesian plane, there is a unique circle passing from P, and with center in the origin $O \equiv (0, 0)$. So, to describe the point P, we can use the radius of such a circle, which is also the distance of P from O, and the angle between thesegment OP with a chosen fixed line, e.g. the $\vec{x}$-axis. Therefore, the pair of values (r, θ), given by radius and angle decribed above, completely and uniquely describes the point P. This description of points in the plane is called the polar coordinate system. Polar coordinates (r, θ) are given by a value $r \geq 0$ and $0 \leq \theta < 2\pi$. When $r = 0$ the point is the origin, and we can think of it as being a circle of zero radius. Here the angle corresponding to it is not uniquely defined.

Of course, we would like to be able to pass from the Cartesian coordinates to polar coordinates when needed. To do this, one uses the same principles that relate cosine, sine and the points on the unit cricle. The only difference is that now the radius is not 1, but r possibly different from 1 (but strictly larger than 0). Given (r, θ) in polar coordinates, we can obtain the Cartesian coordinates by the equations $x = r\cos\theta$ and $y = r\sin\theta$. For the opposite construction, given Cartesian coordinates (x, y), we can obtain the polar coorindates as $r^2 = x^2 + y^2$, and $\theta = \arctan\frac{x}{y}$ if $x \neq 0$, $\theta = \pm\frac{pi}{2}$ when $x = 0$ depending on $y = 1$ or $y = -1$. Observe that for arctan to be defined, one obtains values of θ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, so we can determine the angle θ in $[0, 2\pi)$ after consideration of the signs of x and y.

We now consider the problem of determininng tangents in polar coordinates. A polar curve is given by a function of type $r = f(\theta)$, or more generally through an equation of type $F(r, \theta) = 0$ which we will not consider here. Then, the corresponding Cartesian coordinates x and y are given by $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$. Since both x and y depend on θ alone, we can compute the derivative as follows

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\
&= \frac{\frac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\frac{df}{d\theta}\cos\theta - r\sin\theta} \\
&= \frac{\frac{dr}{d\theta}\sin\theta + f(\theta)\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta},
\end{aligned}
$$

where in the last equality we just rewrote f as r, since f is the function that relates the radius r to θ.

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