LECTURE NOTES MATH 3362 - INTRODUCTION TO COMPLEX VARIABLES

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1. Introduction

Complex numbers are an extension of real numbers, whose precise definition can be given in various different ways. The original reason that led people to this numerical system lies in the fact that real polynomials do not always have real roots. For example, the polynomial $f(x) = x^2 + 1$ does not have any real solution. This is due to the fact that the square of real numbers is always nonnegative, and a hypothetical solution x_0 of $f(x) = 0$ should satisfy $x_0^2 = -1$. It is by extending the real numbers to a new numerical system where such a number exists that we obtain complex numbers. Interestingly, "adding" such solution, called i, to the real numbers substantially allows to have solutions to any nonconstant polynomial equation. This is the content of the celebrated Fundamental Theorem of Algebra.

In this course, we will consider functions from the complex numbers to the complex numbers. The notion of differentiability will be considered in this context, and we will characterize complex differentiability. Power series will be discussed in detail, along with their applications. Cauchy's theorem and integrals over paths will constitute the main bulk of the course. This will lead us to some topological notions that we will discuss, albeit only informally.

Two very well known references upon which this course is based are [1, 2].

2. Complex numbers and complex functions

The complex numbers, usually denoted by $\mathbb{C}$, are a field. This means that they are a set of objects endowed with two binary operations, $+$ and $\cdot$, called addition and multiplication, such that the following axioms hold:

- $+$ and $\cdot$ are associative.
- $+$ and $\cdot$ are commutative.
- There exist distinct neutral elements, $0_{\mathbb{C}}$ and $1_{\mathbb{C}}$, with respect to $+$ and $\cdot$. They are usually simply indicated as 0 and 1. In other words, we have an element 0 such that $x+0 = 0+x = x$ for all $x \in \mathbb{C}$, and an element 1 such that $x \cdot 1 = 1 \cdot x = x$ for all $x \in \mathbb{C}$, with $0 \neq 1$.
- For every $x \in \mathbb{C}$, there exists an additive inverse, called $-x$ such that $x+(-x) = (-x)+x = 0$.
- For every nonzero x, i.e. any $x \neq 0$, there exists a multiplicative inverse, called $\frac{1}{x}$ or x^{-1}, such that $xx^{-1} = x^{-1}x = 1$.
- $\cdot$ distributes over $+$, i.e. we have $x \cdot (y + z) = xy + xz$ for all $x, y, z \in \mathbb{C}$.

Complex numbers are defined as expressions of type $x + iy$, where $i^2 = -1$ and $x, y \in \mathbb{R}$, and such that $+$ and $-$ are defined via the assignments:

- $(x_1 + iy_1) + (x_2 + iy_2) := (x_1 + x_2) + i(y_1 + y_2)$;
- $(x_1 + iy_1) \cdot (x_2 + iy_2) := (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$.

The multiplication symbol $\cdot$ is usually omitted, and juxtaposition of complex numbers simply indicates a product. It is easy to verify that with the definition of $+$ and $\cdot$ above, $\mathbb{C}$ satisfies all the axioms of field as previously claimed. In particular, $0_{\mathbb{C}} = 0 + i0$ and $1_{\mathbb{C}} = 1 + i0$. More generally, $\mathbb{C}$ "contains" the real numbers in the sense that if $x \in \mathbb{R}$, then we can consider the corresponding element $x + i0 \in \mathbb{C}$. Sometimes we will call the map $\mathbb{R} \longrightarrow \mathbb{C}$ given by $x \mapsto x + i0$, the *inclusion morphism* and write it ι. If x and y are two real numbers, then we have that $\iota(x+y) = \iota(x) + \iota(y)$ and $\iota(xy) = \iota(x) \cdot \iota(y)$.

Given a complex number $\alpha = x + iy$, we define the *complex conjugate* of α as $\bar{\alpha} := x - iy$. By direct computation one can see that the complex conjugate has the following remarkable property: $\alpha\bar{\alpha} = x^2 + y^2$. One can also see directly that if $\alpha \neq 0$, then $\alpha^{-1} = \frac{\bar{\alpha}}{x^2+y^2}$.

Theorem 2.1. *Let α and β be complex numbers. Then*

- $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$.
- $\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}$.
- $\bar{\bar{\alpha}} = \alpha$.

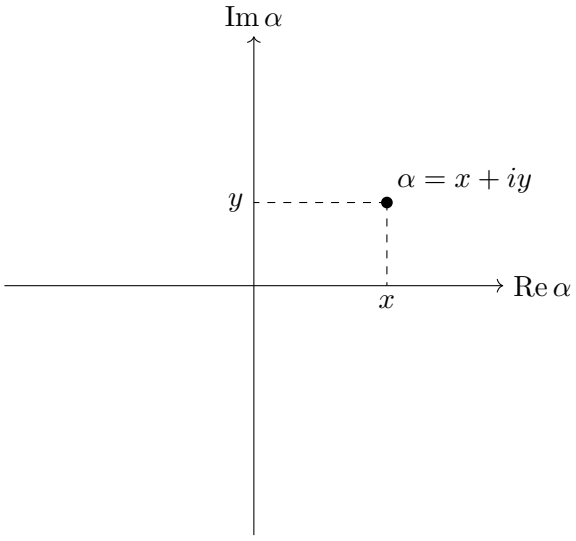
Exercise 2.2. Prove all the previous claims, including the theorem above.

Given a complex number $\alpha = x + iy$, we call x the *real part* of α, written $\mathrm{Re}(\alpha)$, and y is called the *imaginary part* of α, written $\mathrm{Im}(\alpha)$. We define the *absolute value* or *modulus* of α as

$$|\alpha| = \sqrt{x^2 + y^2}.$$

Clearly, from previous observations, one has $|\alpha|^2 = \alpha\bar{\alpha}$.

The complex numbers can be geometrically depicted on the so called complex plane as in the following picture.



Theorem 2.3. *The absolute value of a complex number satisfies the following properties.*

(i) $|\alpha\beta| = |\alpha||\beta|$,
(ii) $|\alpha + \beta| \leq |\alpha| + |\beta|$.

Proof. We have

$$\begin{aligned} |\alpha\beta|^2 &= \alpha\beta\overline{\alpha\beta} \\ &= \alpha\beta\bar{\alpha}\bar{\beta} \\ &= \alpha\bar{\alpha}\beta\bar{\beta} \\ &= |\alpha|^2|\beta|^2. \end{aligned}$$

Taking square roots of the resulting equality gives (i). To prove (ii), first observe that for any $\alpha \in \mathbb{C}$, we have that $\alpha + \bar{\alpha} = 2\,\mathrm{Re}(\alpha)$. Moreover, from the fact that $\bar{\bar{\alpha}} = \alpha$ and $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$, one has the equality $\alpha\bar{\beta} = \overline{\beta\bar{\alpha}}$. Also, note that the real part of a complex number is less than or equal the modulus, and therefore $2\,\mathrm{Re}(\beta\bar{\alpha}) \leq 2|\beta\bar{\alpha}|$. Then, we have

$$\begin{aligned} |\alpha+\beta|^2 &= (\alpha+\beta)\overline{(\alpha+\beta)} \\ &= (\alpha+\beta)(\bar{\alpha}+\bar{\beta}) \\ &= \alpha\bar{\alpha} + \beta\bar{\alpha} + \alpha\bar{\beta} + \beta\bar{\beta} \\ &= |\alpha|^2 + 2\,\mathrm{Re}(\beta\bar{\alpha}) + |\beta|^2 \\ &\leq |\alpha|^2 + 2|\beta\bar{\alpha}| + |\beta|^2 \\ &= |\alpha|^2 + 2|\beta||\bar{\alpha}| + |\beta|^2 \\ &= (|\alpha| + |\beta|)^2, \end{aligned}$$

where we have also used (i). Taking square roots again completes the proof of (ii). □

Definition 2.4. The inequality (ii) in Theorem 2.3 is called the *triangle inequality*.

Remark 2.5. Applying the triangle inequality multiple times, one can also prove that $|\alpha_1 + \cdots + \alpha_n| \leq |\alpha_1| + \cdots + |\alpha_n|$.

Remark 2.6. The equalities $|-\alpha| = |\alpha|$ and $|\alpha| = |\bar{\alpha}|$ also hold.

Let $\alpha = x + iy \in \mathbb{C}$ with $(x, y) \neq (0, 0)$, where $x, y \in \mathbb{R}$. Geometrically, α corresponds to the point (x, y) in the plane, so it has a distance from the origin

$$r = |\alpha| = \sqrt{x^2 + y^2} > 0$$

and it makes an angle θ with the positive real axis. Choosing such an angle θ (an *argument* of α), we have

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

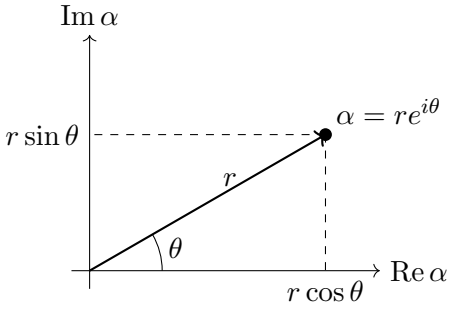
and therefore

$$\alpha = x + iy = r(\cos\theta + i\sin\theta).$$

This representation is called the *polar form* of α. The number r is uniquely determined by α (it is the modulus), while the angle θ is not unique: if θ is an argument of α, then so is $\theta + 2\pi k$ for every $k \in \mathbb{Z}$. We write

$$\mathrm{Arg}(\alpha) = \{\theta + 2\pi k : \ k \in \mathbb{Z}\},$$

and often single out a *principal argument* $\arg(\alpha) \in (-\pi, \pi]$ (or another chosen interval of length 2π) by imposing a convention. The polar form is depicted in the picture below.



Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, polar form can be written more compactly as

$$\alpha = re^{i\theta}.$$

In this notation, multiplication becomes especially simple: if $\alpha = re^{i\theta}$ and $\beta = se^{i\varphi}$, then

$$\alpha\beta = rs\, e^{i(\theta+\varphi)},$$

so moduli multiply and arguments add (modulo 2π). We will prove this in the following.

Theorem 2.7. *Let θ and ϕ be two real numbers, and r and s two nonnegative numbers. Then*

$$re^{i\theta}se^{i\phi} = rse^{i(\theta+\phi)}.$$

Proof. We consider the case of $r = s = 1$, since the proof of the general case is similar. We have that

$$\begin{aligned} e^{i(\theta+\phi)} &= \cos(\theta+\phi) + i\sin(\theta+\phi) \\ &= \cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \sin\phi\cos\theta), \end{aligned}$$

where in the second equality we have used the addition formulas for sine and cosine. Direct computation shows also

$$\begin{aligned} e^{i\theta}e^{i\phi} &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= \cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \sin\phi\cos\theta), \end{aligned}$$

and the equality follows. □

If $\alpha = x + iy$ is a complex number, we define the exponetional $e^{\alpha} := e^{x}e^{iy}$.

Theorem 2.8. *Let α and β be complex numbers. Then*

$$e^{\alpha+\beta} = e^{\alpha}e^{\beta}.$$

Proof. This is an application of the previous result. □

Complex functions are functions $f : S \longrightarrow \mathbb{C}$, where $S \subset \mathbb{C}$. For each $z \in S$, one has $f(z) = u(z) + iv(z)$, and therefore a complex function f is specified once two real-valued functions u and v are given. Since $z = x + iy$ is itself given by specifying two real values, we have that u and v can be seen as two functions $u, v : \mathbb{R}^2 \longrightarrow \mathbb{R}$. So, we can write

$$f(z) = u(x, y) + iv(x, y),$$

where $z = x + iy$.

Example 2.9. Consider the functions $u(x, y) = x^2y$ and $v(x, y) = -3xy$. Then, they define a complex function given by

$$f(x + iy) = u(x, y) + v(x, y) = x^2y - 3ixy.$$

Example 2.10. The most important example of complex function is the power function. For $n \in \mathbb{N}$, we set

$$f(z) = z^n.$$

If $z = re^{i\theta}$, then we have seen that

$$z^n = r^n e^{in\theta} \qquad (n \in \mathbb{N}),$$

which is called De Moivre's formula (in exponential form). Geometrically: raising to the nth power scales the modulus by r^n and multiplies the angle by n. If $D \subset \mathbb{C}$ is the unit disc of $\mathbb{C}$ (i.e. the set of complex numbers with modulus less than or equal to 1), then the map $z \mapsto z^n$ sends D into D. In fact, the same formula above can be used for negative integers, so we can take $n \in \mathbb{Z}$ for negative power functions.

Roots of unity. Let S be the sector of D defined as the set of complex numbers $z = re^{i\theta}$ of the disc such that $\theta \in [0, \frac{2\pi}{n}]$.

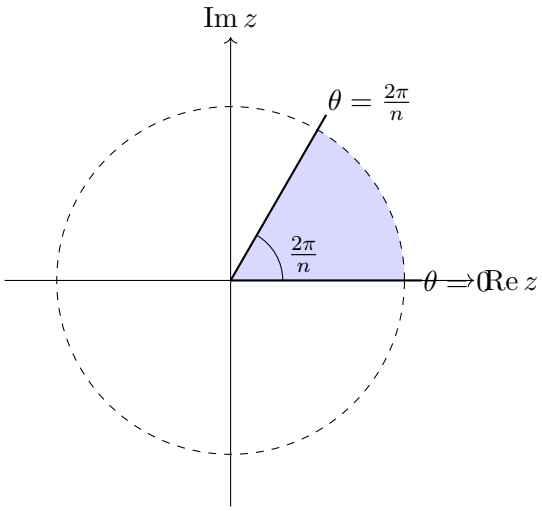


FIGURE 1. Sector of the complex plane defined by $\theta \in [0, 2\pi/n]$.

When considering values of $z \in D$, the unit disc, we have that the real function $r \mapsto r^n$ maps $[0, 1]$ onto $[0, 1]$. The function $\theta \mapsto n\theta$ maps the interval $[0, \frac{2\pi}{n}]$ onto $[0, 2\pi]$. Therefore, we see that $f(z) = z^n$ "wraps" the sector S onto the full disc D.

A *root of unity* is a complex number $\zeta \in \mathbb{C}$ such that $\zeta^n = 1$ for some positive integer n. Since $1 = e^{i2\pi k}$ for every $k \in \mathbb{Z}$, the equation

$$\zeta^n = 1$$

becomes

$$\zeta^n = e^{i2\pi k} \qquad (k \in \mathbb{Z}).$$

Writing $\zeta = e^{i\theta}$ (we must have $|\zeta| = 1$), we obtain

$$e^{in\theta} = e^{i2\pi k},$$

so

$$n\theta = 2\pi k \quad \text{and hence} \quad \theta = \frac{2\pi k}{n}.$$

Therefore, the nth roots of unity are exactly

$$\zeta_k := e^{2\pi ik/n} \qquad (k = 0, 1, \ldots, n-1).$$

They are equally spaced points on the unit circle, forming a regular n-gon. The root $\zeta_1 = e^{2\pi i/n}$ is called a *primitive* nth root of unity; every nth root of unity is a power of ζ_1:

$$\zeta_k = (\zeta_1)^k.$$

All nth roots of a nonzero complex number. If $\alpha = re^{i\theta}$ with $r > 0$, then the solutions of $\beta^n = \alpha$ are

$$\beta_k = r^{1/n} e^{i(\theta+2\pi k)/n}, \qquad k = 0, 1, \ldots, n-1.$$

So: take the nth root of the modulus, and divide the argument by n, producing n equally spaced solutions on the circle of radius $r^{1/n}$.

We need to consider some elementary definitions about the topology of the complex plane. Given a complex number α, an *open disc* of radius $r > 0$ centered at α is the set of complex numbers z such that

$$|\alpha - z| < r.$$

A *closed disc* is defined by the condition

$$|\alpha - z| \leq r.$$

When it is not specified whether a disc is open or closed, we will mean that it is open. This is simply indicated as $D(\alpha, r)$.

We say that a subset $U \subset \mathbb{C}$ is open, if for every $\alpha \in U$, we can find a value $r > 0$ such that $D(\alpha, r) \subset U$. If α is such that $D(\alpha, r) \subset U$ for some $r > 0$, we say that α is an interior point of U. The set of interior points of some subset $A \subset \mathbb{C}$ is denoted U°.

Exercise 2.11. Show that the set of numbers $z \in \mathbb{C}$ such that $\mathrm{Im}(z) > 0$ is open.

Exercise 2.12. Show that a set U is open if and only if every point in U is an interior point.

Let $S \subset \mathbb{C}$. Then, $\alpha \in S$ is said to be a boundary point if $D(\alpha, r) \cap (\mathbb{C} - S) \neq \emptyset$ for all $r > 0$. The set of boundary points of S is denoted by ∂S. The point $\alpha \in \mathbb{C}$ is said to be adherent to S if for every $r > 0$ we have $D(\alpha, r) \cap S \neq \emptyset$. The set $\overline{S}$, called closure of S, is the set containing all adherent points to S. A set S such that $S = \overline{S}$ is said to be closed.

Exercise 2.13. Show that S is closed if and only if $S = \mathbb{C} - U$ where U is open.

Exercise 2.14. Show that $\overline{S} = S \cup \partial S$.

We pose the following important definition.

Definition 2.15. Let f be a complex function defined on S, and let α be an interior point to S. Then, we say that the limit of f as x tends to α exists, and it equals β, in symbols

$$\lim_{z \to \alpha} f(z) = \beta,$$

if $\forall \epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - \beta| < \epsilon$ whenever $|z - \alpha| < \delta$, and $z \in S - \{\alpha\}$.

We will also use the shorthand $f(z) \longrightarrow \beta$ to indicate a limit, and when necessary we will explicitly say that z "tends to" α. In general, we will assume that our functions are defined on open sets, unless otherwise specified. Therefore, for small enough δ, we will always have $|z - \alpha| < \delta$, since any α in the domain of f is an interior point of the domain.

Exercise 2.16. Show that if $\lim_{z\to\alpha} f(z) = \beta$, then $\lim_{z\to\alpha} \overline{f}(z) = \bar{\beta}$.

Exercise 2.17. Use the previous exercise to show that

$$\lim_{z\to\alpha} f(z) = \beta,$$

if and only if

$$\lim_{z\to\alpha} \operatorname{Re} f(z) = \operatorname{Re}\beta, \quad \lim_{z\to\alpha} \operatorname{Im} f(z) = \operatorname{Im}\beta.$$

Definition 2.18. Let α be an element of the domain of f. Then, we say that f is *continuous* at α if

$$\lim_{z\to\alpha} f(z) = f(\alpha).$$

We say that f is a continuous function if it is continuous at each point of its domain.

Example 2.19. The function $f(z) = z^2$ is continuous at every $\alpha \in \mathbb{C}$, since

$$\lim_{z\to\alpha} z^2 = \alpha^2 = f(\alpha).$$

To see this, let $M \geq 2|\alpha| + 1$. Notice that for any given $\epsilon > 0$, we can choose $0 < \delta < \min\{\frac{\epsilon}{M}, 1\}$. Then, if $|z-\alpha| < \delta$, we have $|f(z)-\alpha^2| = |z^2-\alpha^2| = |(z-\alpha)(z+\alpha)| = |z-\alpha||z+\alpha| \leq \delta(|z|+|\alpha|) \leq \delta M < \epsilon$.

Example 2.20. If f and g are two continuous functions, then also $f+g$ and $f\cdot g$ are continuous. If α is a point such that the quotient $f(\alpha)/g(\alpha)$ makes sense, then the function $f(z)/g(z)$ is continuous at α. Lastly, if f is continuous, so are the functions $\operatorname{Re} f$, $\operatorname{Im} f$ and $|f|$.

Example 2.21. If f and g are two continuous functions such that $f(U) \subset V$, where U is the domain of f and V is the domain of g, then $g \circ f$ is continuous.

Exercise 2.22. Show the previous claims.

Definition 2.23. We say that a function is differentiable at the point α if the limit

$$f'(\alpha) = \lim_{z\to\alpha} \frac{f(z) - f(\alpha)}{z - \alpha},$$

exists (finite). Equivalently, we can rewrite the previous limit as

$$f'(\alpha) = \lim_{h\to 0} \frac{f(\alpha + h) - f(\alpha)}{h},$$

As in the case of real functions, if $f : U \longrightarrow \mathbb{C}$ is such that f' is well defined for all $z \in U$, the assignment $z \mapsto f'(z)$ defines a function that we will call the *derivative* of f.

Remark 2.24. Suppose f is a real-valued function of a complex variable, so $f : \mathbb{C} \longrightarrow \mathbb{R} \hookrightarrow \mathbb{C}$ that is differentiable as a complex function at one point α. Then, we have that

$$\lim_{h\to 0} \frac{f(\alpha + h) - f(\alpha)}{h} = \lim_{h\to 0} \frac{f(\alpha + ih) - f(\alpha)}{ih},$$

where h is real. However, since the left-hand-side is in $\mathbb{R}$, and the right-hand-side is in $i\mathbb{R}$ (the imaginary axis), it follows that $f(\alpha) = 0$.

Definition 2.25. We say that a function is *analytic*, or *holomorphic*, if f is differentiable at each point of its domain.

Theorem 2.26. *If f is differentiable at the point α, then it is continuous at α.*

Proof. This fact follows from the equalities

$$\begin{aligned} f(\alpha+h)-f(\alpha) &= h\frac{f(\alpha+h)-f(\alpha)}{h} \\ &= h\cdot f'(\alpha), \end{aligned}$$

which shows that $f(\alpha+h)-f(\alpha)\longrightarrow 0$ as $h\longrightarrow 0$. □

Exercise 2.27. Show that if f is differentiable at α, and $f(z)=u(z)+iv(z)$, then both u and v are continuous.

Exercise 2.28. Show that the "usual" rules for differentiation hold for complex functions too (e.g. Leibniz rule and Chain).

By definition of limit, if we take h to lie in a line in $\mathbb{C}$, the result of $\lim_{h\to 0}\frac{f(\alpha+h)-f(\alpha)}{h}$ does not depend on the chosen line. Therefore, we can choose real values for h and consider the directions along the $\vec{x}$ and $\vec{y}$ axes. In the first case, we have that

$$\lim_{h\to 0}\frac{f(\alpha+h)-f(\alpha)}{h}=\frac{\partial f}{\partial x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}.$$

In the second case we obtain

$$\lim_{h\to 0}\frac{f(\alpha+ih)-f(\alpha)}{ih}=-i\frac{\partial f}{\partial y}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}.$$

Therefore, we find the system of (real) partial differential equations

$$\begin{cases} \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \end{cases}. \tag{1}$$

These are called the Cauchy-Riemann equations.

Exercise 2.29. Show that the partial derivatives in (1) exist as a consequence of the existence of $f'(z)$.

Remark 2.30. Since from (1) $f'(z)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$ it follows that

$$|f'(z)|^2=(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2.$$

Similarly, using again (1) we also find

$$|f'(z)|^2=(\frac{\partial u}{\partial x})^2+(\frac{\partial v}{\partial x})^2,$$

and

$$|f'(z)|^2=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

In particular, we see that $|f'(z)|^2$ is the Jacobian of u and v.

It will be later shown that if f is holomorphic, then f' is also holomorphic. This means that u and v have partial derivatives of all orders, and that the mixed derivatives do not depend on the order of the partial differential operators used. Therefore, applying (1) to this fact we obtain that

$$\begin{aligned} \Delta u &= \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \\ \Delta v &= \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0. \end{aligned}$$

The differential operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the Laplace operator, and functions w that satisfy $\Delta w = 0$ are said to be *harmonic*.

In the assumption that u and v have continuous first-order partial derivatives, we have that

$$\begin{aligned} u(x+h, y+k) - u(x,y) &= \frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k + \epsilon_1, \\ v(x+h, y+k) - v(x,y) &= \frac{\partial v}{\partial x}h + \frac{\partial v}{\partial y}k + \epsilon_2, \end{aligned}$$

where ϵ_1 and ϵ_2 tend to zero faster than $h+ik$. Applying the Cauchy-Riemann equations, we have that

$$f(z+h+ik) - f(z) = (\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})(h+ik) + \epsilon_1 + i\epsilon_2.$$

We therefore obtain that

$$\lim_{h+ik\to 0} \frac{f(z+h+ik) - f(z)}{h+ik} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

We therefore have the following result.

Theorem 2.31. *The function $f = u + iv$ is holomorphic if and only if u and v have continuous first order derivatives that satisfy the Cauchy-Riemann equations.*

3. Topology of metric spaces, sequences and series

Topologies over a set give a notion of "being close" even when the space does not have a numerical quantification for such a term. For instance, for real numbers one can consider the absolute value of the difference of two numbers as the quantification of how far apart they are. Similarly, in $\mathbb{C}$, we can use the modulus of the difference to indicate such notion. On abstract sets, however, it is unclear how such a quantification would be defined. In particular, topologies allow us to define "neighbor" elements, which is of significant importance throughout the sciences.

Definition 3.1. Let X be a set, and $\mathcal{T}$ a collection of subsets of X. Then, we say that $\mathcal{T}$ is a *topology* on X if the following condition are satisfied:

- The empty set $\emptyset$ and the set X are in $\mathcal{T}$.
- Arbitrary union of elements in $\mathcal{T}$ is in $\mathcal{T}$.
- Finite intersection of elements in $\mathcal{T}$ is in $\mathcal{T}$.

Elements in $\mathcal{T}$ are called *open sets*, while complements of open sets are called *closed sets*.

The fundamental idea in topology is that elements that can be separated "easily" by open sets are far apart, while if two points are not easily put in two different open sets are close to each others. Let us consider a very familiar example. We will denote by $\mathcal{P}(X)$ the power set of a given set X, so a topology $\mathcal{T}$ is a subset of the power set $\mathcal{P}(X)$ of X.

Example 3.2. Let X be a set. The simplest topologies on X are the two most obvious ones:

- $\mathcal{T} = \mathcal{P}(X)$.
- $\mathcal{T} = \{\emptyset, X\}$.

It is immediate to show that they are both topologies.

Example 3.3. A less boring topology on a set with 4 elements $X = \{1,2,3,4\}$ is the following, $\mathcal{T} = \{\emptyset, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}, X\}$. One can verify directly that the union of elements in $\mathcal{T}$ is still in $\mathcal{T}$, and that intersections of elements in $\mathcal{T}$ are still in $\mathcal{T}$.

Example 3.4. Consider $\mathbb{R}$ with the following set $\mathcal{T} \subset \mathcal{P}(\mathbb{R})$. We say that A is in $\mathcal{T}$ if for every point $x \in A$, we can find an interval (p, q) containing x and contained in A, $x \in (p, q) \subset A$.

Let us verify that this is indeed a topology. First, observe that $\emptyset$ vacuously satisfies the definition of being in $\mathcal{T}$, and it is therefore open. It is also clear that $\mathbb{R}$ is in $\mathcal{T}$ as well, since any interval is a subset of $\mathbb{R}$, and we can always find an interval containing any chosen element of $\mathbb{R}$. Let $\{A_t\}_{t\in\tau}$ be a family of elements in $\mathcal{T}$, where the index set has arbitrary cardinality (we need arbitrary unions!). We want to show that the union $A = \bigcup_{t\in\tau} A_t$ is in $\mathcal{T}$ as well. Let $x \in A$ be an element of the union. Then, $x \in A_t$ for some $t \in \tau$. Since A_t is open, there is an interval (p, q) containing x such that $(p, q) \subset A_t$. But then (p, q) is an interval contained also in the union A. Therefore A is open, since x was arbitrary. Similar reasoning can be used to show that finite intersections of open sets are still open. We show this for two open sets. The reader then can either directly generalize the argument to arbitrary (but finitely many) open sets, or can use an inductive argument to show that it holds for any (finite) number of open sets. Let A and B be open sets. Then, let $A \cap B$ denote their intersection. If $A \cap B = \emptyset$ there is nothing to show, since we already know that $\emptyset \in \mathcal{T}$. Suppose $x \in A \cap B \neq \emptyset$. Since A is open, we can find $x \in (p_1, q_1) \subset A$. Similarly, since B is open, we can also find $x \in (p_2, q_2) \subset B$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, it follows that we can find $x < q < \min\{q_1, q_2\}$ and $\max\{p_1, p_2\} < p < x$. It is clear that $(p, q) \subset A, B$ and $x \in (p, q)$, showing that $A \cap B$ is open as well.

Therefore, $\mathcal{T}$ is a topology on the real numbers. This topology is called the *Euclidean topology* of $\mathbb{R}$.

Exercise 3.5. Show that if $\{C_j\}_{j\in J}$ is an arbitrary family of closed sets, then $\bigcap_{j\in J} C_j$ is closed. So, while only finite intersections of open sets are open, arbitrary intersections of closed sets are closed.

What can you say about unions of closed sets?

There is a general way of defininining a topology by listing a set of subsets with some additional properties.

Theorem 3.6. *Let X be a set and let $\mathcal{B}$ be a family of subsets of X. Then, the set of all unions of elements of $\mathcal{B}$ defines a topology on X if and only if $\mathcal{B}$ satisfies the following two properties:*

(1) $\bigcup_{B\in\mathcal{B}} B = X$.
(2) *If B_1 and B_2 are elements in $\mathcal{B}$, and $x \in B_1 \cap B_2$, then there is an element $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.*

Proof. Suppose first that $\mathcal{T}$, defined as the set of all unions of elements of $\mathcal{B}$ is a topology. We want to verify that (1) and (2) hold. Since X is always an element of a topology, it means that $X \in \mathcal{T}$, which means that X can be written as the union of elements in $\mathcal{B}$. This shows (1). To show (2), since finite intersection of elements in $\mathcal{T}$ is in $\mathcal{T}$, and since $\mathcal{B} \subset \mathcal{T}$, we find that $B_1 \cap B_2 \in \mathcal{T}$. This means that $B_1 \cap B_2 = \cup_{t\in\tau} B_t$ for some index set τ and elements B_t of $\mathcal{B}$. Given $x \in B_1 \cap B_2$, there must exist some B_q such that $x \in B_q$. Clearly $B_q \subset \cup_{t\in\tau} B_t = B_1 \cap B_2$, which shows (2).

Conversely, let us now consider a family $\mathcal{B}$ satisfying both (1) and (2). Defining $\mathcal{T}$ to be the family of all unions of elements of $\mathcal{B}$, we want to verify that such $\mathcal{T}$ is indeed a topology. Observe that $X \in \mathcal{T}$ by (1), and also $\emptyset \in \mathcal{T}$ since it is the empty union of elements from $\mathcal{B}$. It is clear that the union of A_t, with $t \in \tau$ some index set, and A_t in $\mathcal{T}$ results in an element of $\mathcal{T}$. We need to verify that the intersection of two open sets is still open. Observe that from (2), we find that for

any $B_1, B_2 \in \mathcal{B}$, we have the equality

$$B_1 \cap B_2 = \bigcup \{C \in \mathcal{B} \mid C \subset B_1 \cap B_2\}. \tag{2}$$

If A_1, A_2 are elements of $\mathcal{T}$, then by definition we have $A_1 = \cup_i U_i$ and $A_2 = \cup_i V_j$ for open sets U_i and V_j. Then, we have that

$$A_1 \cap A_2 = \bigcup_{i,j} U_i \cap V_j = \bigcup \{C \in \mathcal{B} \mid C \subset U_i \cap V_j\}. \tag{3}$$

This completes the proof. □

Definition 3.7. A family $\mathcal{B}$ satisfying the two properties of the previous theorem therefore induces a topology $\mathcal{T}$. The family $\mathcal{B}$ is then called a basis for $\mathcal{T}$.

Exercise 3.8. Show that the set of open discs in $\mathbb{C}$ satisfies the requirements of the previous theorem, and therefore it induces a well defined topology on $\mathbb{C}$.

Remark 3.9. Given a subset Y of a topological space X, there is a topology automatically defined on Y, where the open sets are given by the collection of intersections of open sets of X with Y. When we refer to a subspace of a topological space, we will mean this topology.

Definition 3.10. Let X be a topological space, and $B \subset X$. We then define the following concepts.

(1) The set obtained by taking the union of all open sets contained in B, denoted by B° and called the *interior* of B.
(2) The set obtained as the intersection of all closed sets containing B, denoted by $\bar{B}$, and called the *closure* of B.
(3) The set $\partial B := \bar{B} - B^\circ$, called the *boundary* of B.
(4) An element x is said to be an accumulation point, or limit point, of $E \subset X$ if for any open set V containing x we can find an element $y \neq x$ with $y \in E$.

The elements of B° are also called *interior points* of B.

Exercise 3.11. Show that the previous definition coincides with the corresponding ones given for $\mathbb{C}$ in Section 2 where the topology of $\mathbb{C}$ is the one of Exercise 3.8.

Definition 3.12. A subset A of a topological space X is said to be *dense* in X if $\bar{A} = X$.

The fundamental idea of using topological spaces, is that it allows one to define continuous functions in a very general way. Here we will also call functions simply as *maps*, to indicate that the domain and target are arbitrary (not necessarily numerical sets).

Definition 3.13. Let X, Y be topological spaces, and let $f : X \longrightarrow Y$ be a map between them. Then, we say that f is *continuous* if $f^{-1}(V)$ is an open set of X whenever V is an open set of Y. In other words, the preimage of open sets is open.

If $f : X \longrightarrow Y$ is a continuous map having a continuous inverse $f^{-1} : Y \longrightarrow X$, we say that f is a *homeomorphism*.

Remark 3.14. Observe that continuity is equivalently defined as $f^{-1}(C)$ is closed in X whenever C is a closed subset of Y. This is shown by considering the complements.

Exercise 3.15. Show that continuity given in terms of open sets coincides with the notion of continuity in terms of limits as previously given.

Definition 3.16. Let X be a set. Then, a *metric*, or *distance* on X, is a map $d : X \times X \longrightarrow \mathbb{R}$ satisfying the following conditions.

(1) $d(x, y) \geq 0$ for all x, y, and $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$ for all x, y.
(3) $d(x, z) \leq d(x, z) + d(y, z)$ for all x, y, z.

Condition (3) is called the *traingule inequality*. A set X endowed with a metric d is called a *metric space*, and often denoted as a pair (X, d).

Example 3.17. Let X be a set. Define the following map

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

Then, d defines a metric on X.

Example 3.18. Let $\mathbb{C}$ be the complex numbers and define the function $d(x, y) = |x - y|$. Then, d is a metric on $\mathbb{C}$.

Example 3.19. An example of a metric defined on a space of functions is the following. Consider $C([a, b])$, the set of real-valuedcontinuous functions defined on the interval $[a, b]$. Then, we define $d(f, g) = \max_{x\in[a,b]} |f(x) - g(x)|$. We know that this maximum exists since continuous functions admit a maximum over intervals.

Definition 3.20. Given a metric space (X, d), and an element $x \in X$ and a real number $r > 0$, we define the *ball of radius* r around x as the set $B(x, r) := \{y \in X \mid d(x, y) < r\}$.

Definition 3.21. A subset A of a metric space X is said to be a neighborhood of x if $B(x, r) \subset A$ for some $r > 0$.

Theorem 3.22. *Given a metric space (X, d), let $\mathcal{B}$ be the family of balls $B(x, r)$ for $x \in X$ and $r > 0$. Then $\mathcal{B}$ satisfies the properties of Theorem 3.6.*

Proof. It is clear that $\bigcup B \in \mathcal{B} B = X$, since each x belongs to the ball $B(x, 1)$, which is an element of the family $\mathcal{B}$. Let $y \in B(x_1, r_1) \cap B(x_2, r_2)$. This means that $d(y, x_i) < r_i$ for $i = 1, 2$. Then, we choose $0 < \rho_i < r_i - d(y, x_i)$ and set $\rho := \min\{\rho_1, \rho_2\}$. Let $z \in B(y, \rho)$. Then $d(x_i, z) \leq d(x_i, y) + d(y, z) < d(x_i, y) + \rho_i < r_i$, which means that $z \in B(x_i, r_i)$, and therefore $z \in B(x_1, r_1) \cap B(x_2, r_2)$. In other words, we have found $B(y, \rho) \subset B(x_1, r_1) \cap B(x_2, r_2)$. □

Example 3.23. The topology we use for $\mathbb{C}$ is the one having the discs as a basis, by the previous theorem. This coincides with the topology given in Exercise 3.8. So, the construction of topologies by taking unions of balls is a general definition for metric spaces, and the case of complex numbers is an example of it.

Definition 3.24. Let $E \subset X$ be a asubset of the topological space X. Then, an *open cover* of E is a family of open sets $\{G_\alpha\}$ such that $E \subset \bigcup_\alpha G_\alpha$.

Definition 3.25. A subset K of the topological space K is said to be *compact* if for any open cover $\{G_\alpha\}$ of E, there is a finite subcover, i.e. we can find finitely many indices $\alpha_1, \ldots, \alpha_n$ such that $E \subset \bigcup_{i=1}^n G_{\alpha_i}$.

Remark 3.26. Compact sets play a fundamental role in Analysis. They give us a pathway to pass from infintely many objects to only finitely many.

Theorem 3.27. *Let $f : X \longrightarrow Y$ be a continuous function, and let X be compact. Then, $f(X)$ is compact in Y.*

Proof. Let $\{V_\alpha\}$ be a collection of open sets of Y covering $f(X)$. Then, since f is continuous, every preimage $U_\alpha := f^{-1}(V_\alpha)$ is open in X. Moreover, the U_α's cover X. Since X is compact, we can find a finite subcover. The image of this subcover consists of finitely many V_α's that cover $f(X)$, showing that $f(X)$ is also compact. □

Theorem 3.28. *Compact subsets of metric spaces are closed.*

Proof. Let K be compact in X, metric space. We want to show that $X - K$ is open (which would mean that K is closed by definition). Let $x \in X - K$, and for every $y \in K$ consider the ball around y of radius $\rho_y := \frac{d(x,y)}{2}$, $B(y, \rho_y)$. For any chosen y, $\rho_y < d(x, y)$, therefore x is not in $B(y, \rho_y)$. Since each y belongs to one of the balls – it is certainly in the ball $B(y, \rho_y)$ of which it is center – the family of balls covers K: $K \subset \bigcup_y B(y, \rho_y)$. Being compact, K can be covered with only finitely many of them, so we can find $y_1, \ldots, y_n$ such that $K \subset \bigcup_{i=1}^n B(y_i, \rho_{y_i})$. Of course, we find that $A = \bigcup_{i=1}^n B(y_i, \rho_{y_i})$ is an open set, being union of open sets. Also, by construction x is not in A (because it is not in any of the balls!). We can also say more, if $0 < r < \min\{\rho_{y_i}\}$, $B(x, r)$ does not intersect any of the balls $B(y_i, \rho_{y_i})$, and therefore it does not intersect A. This means that x is an interior point to $X - K$. Since we can do this for any $x \in X - K$, it means that $X - K$ consists only of interior points, which means it is open. This completes the proof. □

Definition 3.29. Given a subset Y of a metric space (X, d), we say that Y is bounded if we can find $\rho > 0$ such that $Y \subset B(x, \rho)$ for some $x \in X$.

Theorem 3.30. *Compact subsets of metric spaces are bounded.*

Proof. Let K be a compact subset of the metric space X. Let x^* be a point of X. Then, consider the family $\{B(x^*, \rho) \mid \rho > 0\}$. This family covers K, and since K is compact, we can find a finite subcovering, meaning that $K \subset \bigcup_{i=1}^n B(x^*, \rho_i)$. Choosing $\rho \geq \max_i\{\rho_i\}$, it follows that $K \subset B(x^*, \rho)$. □

Theorem 3.31. *Closed subsets of compact sets are compact.*

Proof. Let $F \subset K$ be closed in X, and let $K \subset X$ be compact. Let $\{V_\alpha\}$ be an open cover of F. Since F is closed, $A = X - F$ is open, and $\{V_\alpha\} \cup \{A\}$ is an open cover of K. So, we can find an open subcover, i.e. we can find finitely many α_i such that $K \subset \bigcup_{i=1}^n V_{\alpha_i} \bigcup A$. Since $F \subset K$, we also have $F \subset \bigcup_{i=1}^n V_{\alpha_i} \bigcup A$. But $A = X - F$, so A does not contribute to the previous inclusion and we can remove it without changing it. We have $F \subset \bigcup_{i=1}^n V_{\alpha_i}$, but this means that we have found an open subcover of F from $\{V_\alpha\}$. So, F is compact. □

Corollary 3.32. *If F is closed, and K is compact, then $F \cap K$ is compact.*

Proof. Since K is compact, it is also closed by Theorem 3.28. We have that $F \cap K$ is closed as well, being the intersection of two closed sets. Since $F \cap K \subset K$ is a closed subset of a compact, by Theorem 3.31 it is also compact. □

Theorem 3.33. *If $\{K_\alpha\}$ is a collection of compact subsets of X with the property that any intersection of a finite subcollection of K_α's is nonempty. Then $\bigcap_\alpha K_\alpha \neq \emptyset$ as well.*

Proof. We fix a member K_* of $\{K_\alpha\}$. Set $G_\alpha = X - K_\alpha$ for all $\alpha \neq *$. Suppose that K_* does not intersect every K_α. Then, it means that every element of K_* belongs to some G_α. In other

words, $K_* \subset \bigcup_{\alpha \neq *} G_\alpha$. Since K_* is compact, we can find finitely many α's, say $\alpha_1, \ldots, \alpha_n$ such that $K_* \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$. However, this means that $K_* \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \emptyset$, against the hypothesis that every choice of finitely many of K_α have nonempty intersection. This contradiction proves the result. □

The following corollary is very useful.

Corollary 3.34. *If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_{n+1} \subset K_n$ for all n, then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.*

Theorem 3.35. *If E is an infinite subset of a compact set K, then E has an accumulation point in K.*

Proof. If no point of K is accumulation point of E, for any choice of $x \in K$ we can find a neighborhood V_x of x which intersects E at most in a single point (which is x itself if $x \in E$). Since E is infinite, no finite subcollection of $\{V_x\}$ can cover E, and therefore no finite subcollection can cover K, against the fact that K is compact. So, K must have some accumulation point of E. □

Definition 3.36. Let X be a set. A sequence in X is a function $f : \mathbb{N} \longrightarrow X$. Sequences are usually denoted as $\{x_n\}_{n \in \mathbb{N}}$ or simply $\{x_n\}$, where $f(n) = x_n$. Given a sequence, we call a subsequence some function $f : \mathbb{N} \longrightarrow X$ composed with a monotonic strictly increasing function $q : \mathbb{N} \longrightarrow \mathbb{N}$, i.e. $f \circ q$. This means, in practice that if $\{x_n\}$ is a sequence, a subsequence is some $\{x_{n_k}\}$ where n_k are a subset of n which is strictly increasing with k.

Definition 3.37. Let X be a topological space, and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ converges to $x \in X$ if for each neighborhood U of x, there exists a natural number $\nu \in \mathbb{N}$ such that $x_n \in U$ for every $n > \nu$, $n \in \mathbb{N}$.

Exercise 3.38. Prove the following. If (X, d) is a metric space, then $\{x_n\}$ converges to x if and only if for each $\epsilon > 0$ we can find $\nu \in \mathbb{N}$ such that $d(x, x_n) < \epsilon$ for each $n > \nu$.

Definition 3.39. Let (X, d) be a metric space. We say that a sequence $\{x_n\}$ is a *Cauchy sequence* if for each $\epsilon > 0$ there exists $\nu \in \mathbb{N}$ such that $n, m > \nu$ implies $d(x_n, x_m) < \epsilon$.

Definition 3.40. A metric space (X, d) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges in X.

Theorem 3.41. *A compact metric space is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in the compact metric space (X, d). If $\{x_n\}$ is finite (i.e. the values x_n only attain finitely many different values in X), it is clear that x_n converges in X, because if that were not the case it could not be Cauchy. We can therefore assume that $\{x_n\}$ is infinite. In this case, applying Theorem 3.35, we find that $\{x_n\}$ has an accumulation point $x^* \in X$. We want to show that x^* is the limit of $\{x_n\}$. Let $\epsilon > 0$. Since $\{x_n\}$ is Cauchy, there exists ν such that if $n, m > \nu$, then $d(x_n, x_m) < \frac{\epsilon}{2}$. Since x^* is an accumulation point, it follows that there exists some $r > \nu$ such that $x_r \in B(x, \frac{\epsilon}{2})$. Then, if $n > \nu$, it follows that $d(x^*, x_n) \leq d(x^*, x_r) + d(x_r, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, which shows that $x_n \longrightarrow x^*$. □

Definition 3.42. A subset Y of a metric space (X, d) is said to be *totally bounded* if for any choice of $\epsilon > 0$, we can cover Y by finitely many balls of radius ϵ.

Exercise 3.43. Prove that a subset of a totally bounded set is totally bounded itself.

Definition 3.44. A subset K of a metric space (X, d) is said to be sequentially compact, if any sequence in K admits a convergent subsequence in K.

Theorem 3.45. *A subset of a metric space is compact if and only if it is complete and totally bounded.*

Proof. Let K be a compact subset of the metric space X. We have seen that K must be complete by Theorem 3.41. Since a compact subset is clearly totally bounded, the first implication follows.

Conversely, suppose K is complete and totally bounded. Assume that there exists a covering of K, say $\{A_i\}_{i\in I}$ which does not admit a finite subcovering. Since K is totally bounded, we can find finitely many elements $x_1, \ldots, x_r$ of K such that $K \subset B(y_1, \frac{1}{2}) \cup \cdots \cup B(y_r, \frac{1}{2})$. If each ball $B(y_j, \frac{1}{2})$ could be covered by finitely many of the opens A_i, it would follow that also K could be covered by finitely many of the A_i. Therefore, there must be at least one ball that cannot be covered by finitely many A_i's. Rename this ball $B(x_1, \frac{1}{2})$. Since $B(x_1, \frac{1}{2}) \cap K$ is a subset of a totally bounded set, it follows that it is itself totally bounded. Therefore, we can cover it with balls of radius $\frac{1}{2^2}$. As before, we must be able to find a ball which cannot be covered by finitely many A_i's. Call this ball $B(x_2, \frac{1}{2^2})$. We can proceed in this way by constructing at each step a ball that is not covered by finitely many of the A_i's, which has radius $\frac{1}{2^n}$. So, we have a sequence $B(x_n, \frac{1}{2^n})$. Observe that each x_{n+1} belongs to $B(x_n, \frac{1}{2^n}) \cap K$. Therefore, we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &< \frac{1}{2^n} + \cdots + \frac{1}{2^{n+p-1}} \\ &\leq \frac{1}{2^{n-1}}. \end{aligned}$$

This means that $\{x_n\}$ is a Cauchy sequence in K. Since K is complete, it must converge to some $x^* \in K$. Since $\{A_i\}$ is a covering, there exists an open A_* which contains x^*. By definition of open set, there exists a $\delta > 0$ such that $B(x^*, \delta) \subset A_*$. Choose ν large enough so that $d(x_n, x^*) < \delta/2$ and also $\frac{1}{2^n} < \delta/2$ whenever $n > \nu$. Then, we have that $B(x_n, \frac{1}{2^n}) \subset B(x^*, \delta) \subset A_*$. But this contradicts our construction of $B(x_n, \frac{1}{2^n})$. This contradiction shows that K is compact. □

Theorem 3.46. *A subset K of a metric space is compact if and only if it is sequentially compact.*

Proof. Assume that K is compact, and let $\{x_n\}$ be a sequence in K. If $\{x_n\}$ attains only finitely many different values, then there is a constant subsequence, which clearly is convergent. Therefore, we can assume that $\{x_n\}$ is infinite. Then, applying Theorem 3.35 we find that there exists an accumulation point $x^* \in K$ for the sequence $\{x_n\}$. For each $k \in \mathbb{N}$, consider $B(x^*, 1/k)$. Since x^* is an accumulation point of $\{x_n\}$ we can find for each k an element $x_{n_k} \in B(x^*, 1/k)$. The subsequence x_{n_k} converges to x^* and this completes the first part of the proof.

Conversely, assume that K is sequentially compact. By contradiction suppose that K is not compact. By Theorem 3.45 it follows that K is not totally bounded, since that sequential compactness implies completeness (prove it!). Therefore, there exists an $\epsilon > 0$ such that K cannot be covered by finitely many balls of radius ϵ. Pick x_1 arbitrarily. Since K cannot be covered by finitely many $B(x, \epsilon)$, it follows that $K - B(x_1, \epsilon)$ is not empty. Pick x_2 in K which is not in $B(x_1, \epsilon)$. Since K is not covered by finitely many ϵ balls, we can find x_3 in K that does not lie in $B(x_1, \epsilon) \cup B(x_2, \epsilon)$. We can continue this construction inductively so that at each n we find x_{n+1} in K such that x_{n+1} is not in $B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon)$. The resulting sequence satisfies the property that $d(x_n, x_m) \geq \epsilon$ for every n, m. Therefore, it does not admit any convergent subsequence, against the fact that K is sequentially compact. This contradiction completes the proof. □

Theorem 3.47. *A subset K of $\mathbb{C}$ is compact if and only if it is closed and bounded.*

Proof. By using the previous results, the only thing that one needs to prove is that bounded subsets of $\mathbb{C}$ are totally bounded. This is done by considering a ball that contains a bounded subset K. Then, this ball is contained in a square (product of intervals). We can subdivide these squares in subsquares which are arbitrarily small. Then, we can consider discs that cover these squares where the radii are of some appropriate size (since the squares can be made arbitrarily small). If any of the balls are centered outside of the subset K, the reader should verify that a suitable choice can be made in such a way that new balls centered in K can be chosen, therefore completing the proof. □

Exercise 3.48. Give a similar characterization of compacts in $\mathbb{R}$, and prove the result. Use the version of the result of $\mathbb{R}$ to show that if $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function, then f attains its maximum and minimum on $[a, b]$. (This is the well known Extreme Value Theorem from calculus.)

State and prove an analogous version for maps $f : K \longrightarrow \mathbb{R}$ from compact metric spaces to the real numbers.

We now introduce the notion of connectedness, which will also be used in the rest of the course.

Definition 3.49. A topological space X is said to be *connected* if the only subsets of X that are both open and closed are X and $\emptyset$.

Remark 3.50. Using the topology of subspaces discussed before, one can formulate connectedness also for a subset of a topological space by adapting the previous definition. Moreover, we can give the following characterization of connectedness: A subset A of a topological space X is connected if and only if there is no pair of open subsets U and V of X, such that $A \subset U \cup V$, $A \cap U \cap V = \emptyset$, and both $U \cap A$ and $V \cap A$ are nonempty.

Exercise 3.51. Prove the characterization of connectedness given above.

Lemma 3.52. *The following are equivalent:*

- *X is not connected.*
- *X is disjoint union of two proper open subsets.*
- *X is disjoint union of two proper closed subsets.*

Proof. Left to the reader as a very useful exercise. □

Theorem 3.53. *Let $f : X \longrightarrow Y$ be a continuous function. Then, if X is connected, $f(X)$ is connected as well.*

Proof. Suppose that $Z \subset f(X)$ is a nonempty subset which is both open and closed. This means that $Z = f(X) \cap A$ for some open A, and $Z = f(X) \cap C$ for some closed C, since $f(X)$ is a subspace of Y. By continuity, $f^{-1}(Z) = f^{-1}(A)$ is open in X, and $f^{-1}(Z) = f^{-1}(C)$ is closed in X. Therefore, $f^{-1}(Z)$ is a set that is both open and closed in the connected X, which means that $f^{-1}(Z) = X$, and therefore $Z = f(X)$. Since we have showed that any nonempty subset of $f(X)$ that is both open and closed is $f(X)$ itself, it follows that $f(X)$ is connected. □

Theorem 3.54. *Any interval $[a, b]$ of $\mathbb{R}$ is connected (in the Euclidean topology).*

Proof. We will prove it for $[0, 1]$, since the same proof works for any choice of a and b. Suppose that C, D are two nonempty closed subspaces of $[0, 1]$ with $C \cup D = [0, 1]$. We need to show that $C \cap D \neq \emptyset$, which would complete the proof by Lemma 3.52. Let $a_1 \in C$ and $b_1 \in D$, which can

be found since C and D are both nonempty. We can assume that $a_1 < b_1$, or otherwise we can just rename them. The interval (a_1, b_1) can be bisected. One of the two halves, will have its left end point in C and right end point in D, since C and D are closed. Therefore, we denote this subinterval by (a_2, b_2) and continue the process by bisecting it. Proceeding inductively, we obtain two sequences a_n and b_n of points in $[0, 1]$ such that at each step $a_n \in C$ and $b_n \in D$. Since $|b_n - a_n| \longrightarrow 0$, it follows that the two sequences have a common point, say c. Since C and D are closed, c belongs to both C and D, therefore completing the proof. □

Exercise 3.55. Adapt the previous proof to the case of intervals that can be also half infinite, or the whole real line $\mathbb{R}$, i.e. subsets such as $(a, +\infty)$, $(-\infty, a)$ or $\mathbb{R} = (-\infty, +\infty)$.

Let E be a subset of the reals $\mathbb{R}$. We call a number α a lower bound for E if $\alpha \leq x$ for all $x \in E$. Let A be the set of lower bounds of E. Consider the complement of A, i.e. the set $\mathbb{R} - A$. If $x \in \mathbb{R} - A$, it means that x is not a lower bound of E. Therefore, there exisits some $y \in E$ such that $y < x$. This means that the open set $(x - \epsilon, x + \epsilon)$ is in $\mathbb{R} - A$ whenever $0 < \epsilon < x - y$. In other words, it follows that $\mathbb{R} - A$ is open, and therefore A is closed. Moreover, if A does not contain any largest element, it is also open. Since the real line is connected by Exercise 3.55, we have a total of three possibilities:

- $A = \emptyset$;
- $A = \mathbb{R}$;
- A contains a largest number.

The cases $A = \emptyset$ and $A = \mathbb{R}$ correspond to the case where A does not have a largest element, and therefore must be both open and closed, and by connectedness of $\mathbb{R}$, the only possibilites are the first two. The third one is possible since A does not need to be open if it has a largest element. So, we have found that if E is a subset of the real line, we have that the set of lower bounds could be either empty, or the whole real line, or it has a maximum element. Such maximum element of lower bounds is called *greatest lower bound*, or *infimum*, and denoted $\inf E$. With the convention that $\inf E = -\infty$ if $A = \emptyset$, and $\inf E = +\infty$ if $A = \mathbb{R}$, we have that any subset of the real line admits an infimum. Clearly, $\inf E = +\infty$ if and only if $E = \emptyset$.

A similar discussion shows that any subset E of the real line admits a *smallest upper bound*, or *supremum*, which is the smallest of all upper bounds, with the convention that $\sup E = +\infty$ if the upper bounds are the empty set, and $\sup E = -\infty$ if the upper bounds are the whole real line. Again, as before, $\sup E = -\infty$ if and only if $E = \emptyset$. If $\inf E$ is an element of E, then we will call it $\min E$, and if $\sup E$ is in E, we will denote it $\max E$.

We now prove the following.

Theorem 3.56. *If E is a connected nonempty subset of the real line, then it is an interval (possibly infinite on one or both sides).*

Proof. Let $a = \inf E$ and $b = \sup E$, where E is a nonempty connected subset of $\mathbb{R}$. Clearly, $E \subset [a, b]$ or $E \subset (a, b)$, or semiopen on the left or right, where we use the convention that if $a = -\infty$ or $b = +\infty$, then the interval is left or right semiopen. Suppose that there exists an element $\zeta \in (a, b)$ such that $\zeta \notin E$. Then the open sets $(-\infty, \zeta)$ and $(\zeta, +\infty)$ contain E, but since E is connected, one of the two intersections $(-\infty, \zeta) \cap E$ or $(\zeta, +\infty) \cap E$ must be empty, or we would find two disjoint open sets whose union contains E (remember the characterization of connectedness in Remark 3.50). Suppose then, without loss of generality, that $(-\infty, \zeta) \cap E = \emptyset$. It follows that $\zeta \leq x$ for all $x \in E$, which means that ζ is a lower bound of E, which is not possible

since we had chosen $\zeta > a = \inf E$. Similarly, it cannot be possible to have $(\zeta, +\infty) \cap E = \emptyset$ without incurring in a contradiction. It follows that $E = [\inf E, \sup E]$, or $E = (\inf E, \sup E)$, or semiopen on the left or right (again with the convention that if either of them are infinite, we mean semiopenness of the interval). The proof is complete. □

Corollary 3.57. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Then, for any value y where $q < y < p$ $q, p \in f([a, b])$, there exists $c \in [a, b]$ such that $f(c) = y$.*

Proof. By Theorem 3.53 and Theorem 3.54 we find that the image of f, $f([a, b])$, is connected in $\mathbb{R}$. By Theorem 3.56 it follows that $f([a, b])$ is an interval, and therefore the result follows. □

Exercise 3.58. Use the previous corollary along with the extreme value theorem that you proved in Exercise 3.48 to prove the Intermediate Value Theorem, i.e. that a continuous function f on $[a, b]$ attains any value between $\min f$ and $\max f$.

Definition 3.59. A nonempty connected open set in $\mathbb{C}$ is called a *region*. The closure of a region is called a *closed region*.

In the following, by polygonal curve we will mean a curve in $\mathbb{C}$ which consists of the union of finitely many line segments. The next result characterizes the regions of $\mathbb{C}$.

Theorem 3.60. *A nonempty open set $A \subset \mathbb{R}^n$ or $A \subset \mathbb{C}^n$ is connected if and only if any two points in A can be joined through a polygonal curve.*

Proof. We will prove the result in the case of $\mathbb{C}$, but the same proof can be adapted to $\mathbb{R}^n$ or $\mathbb{C}^n$.

Let A be open and connected and let $a \in A$. We define two subsets of A: A_1 will indicate the set of points of A that can be connected to a via a polygon, and we will let A_2 indicate the set of points of A which are not in A_1. We want to prove that both A_1 and A_2 are open. Since A is open, if $b \in A$, b is interior to A, and therefore there exists $r > 0$ such that the disc $D(b, r) \subset A$. Any point in $x \in D(b, r)$ can be joined with b via a segment $L = \lambda b + (1 - \lambda)x$ where $\lambda \in [0, 1]$. We have that for each $\lambda \in [0, 1]$, $|\lambda b + (1 - \lambda)x - b| = |(\lambda - 1)b + (1 - \lambda)x| = |\lambda - 1||b - x| \leq |b - x| < r$, so that the whole segment L lies inside $D(b, r) \subset A$. Therefore, the whole disc $D(b, r)$ is a subset of A that can be connected to b via a line segment. If $b \in A_1$, in particular, there is a polygon connecting b to a, and for any point x in a disc $D(b, r)$ as found above, we can connect x to b via a segment, and then b to a via a polygon, which means that x can be connected to a via a polygon. In other words, $D(b, r) \subset A_1$. So, all points of A_1 are interior points, and therefore A_1 is open. Suppoe now $b \in A_2$. As before, let $D(b, r)$ be a disc contained in A. Then, if any point $x \in D(b, r)$ were in A_1, it could be joined to a via a polygonal curve. Since x can be joined to b via a line segment lying in A, it would follow that b would be an element of A_1, contrary to the fact that b was chosen in A_2. Therefore, $D(b, r) \subset A_2$, which shows that A_2 is open as well. Since A_1 and A_2 are disjoint, open, and cover A, the fact that A is connected implies that at least one of them is empty. Since A_1 contains the point a at the very least (the constant curve is a polygonal curve), it follows that A_2 is empty. Therefore, all points of A can be joined to a via a polygonal curve, and any two points of A can be joined via a polygonal curve which passes through a. This proves the first implication.

Conversely, suppose that any two points in A can be joined via a polygonal curve. Suppose that $A = A_1 \cup A_2$ where A_1 and A_2 are open, nonempty, and disjoint. Choose $a_1 \in A_1$ and $a_2 \in A_2$. By assumption, we can find a polygonal curve connecting a_1 and a_2. At least one of the segments in the polygon will join a point of A_1 to a point of A_2, since the polygon starts in A_1 and ends in A_2.

Therefore, we can restrict ourselves to the case where a_1 and a_2 are joined via a single segment, without loss of generality. This segment is written as $z_\lambda = (1-\lambda a_1) + \lambda a_2$, where $\lambda \in [0,1]$. Define the two sets Λ_1 and Λ_2 as follows. $\Lambda_i = \{\lambda \in [0,1] \mid z_\lambda \in A_i\}$, for $i = 1, 2$. It is clear that Λ_i are open, disjoint, and their union is the whole $[0,1]$. Moreover, $0 \in \Lambda_1$ and $1 \in \Lambda_2$. Since $[0,1]$ is connected, this is not possible, and this contradition shows that A is connected too. □

Definition 3.61. Let A be a subset of $\mathbb{C}^n$ (or $\mathbb{R}^n$). Then, we say that the subset B of A is a *component* of A if B is connected, and there is no proper set containing B which is also connected.

Theorem 3.62. *Every set A has a unique decomposition into components.*

Proof. Let $a \in A$, and define the set $C(a)$ as the union of all connected subsets of A that contain a. It is clear that $C(a)$ is not empty, since the singleton $\{a\}$ is connected, and therefore $a \in C(a)$. Tt is clear that if $C(a)$ is connected, it is a component, since it is maximal among the connected sets containing a, so any other set containing $C(a)$ would coincide with $C(a)$. Moreover, in the assumption that each $C(a)$ is connected, if $C(a)$ and $C(b)$ are two such sets, only two possibilities can occur: they are either disjoint, or they coincide. In fact, suppose that $c \in C(a) \cap C(b)$. Then, since $C(a)$ is connected and c is an element of it, we have that $C(c) \subset C(a)$, which implies $C(a) = C(c)$ by maximality of $C(c)$. Similarly, $C(b) = C(c)$ and therefore $C(a) = C(b)$.

We therefore need only to show that each $C(a)$ is connected to complete the proof. Suppose by way of contradiction otherwise. Then, let U_1 and U_2 be two open sets such that $C(a) = U_1 \cup U_2$ where U_1 and U_2 do not intersect inside $C(a)$, but such that $C(a) \cap U_1, C(a) \cap U_2 \neq \emptyset$. The element a lies in one of U_1 and U_2, and without loss of generality we can assume that $a \in U_1$. Since U_2 intersects $C(a)$ nontrivially, we have that $b \in C(a)$ for some $b \in C(a)$. By definition of $C(a)$, we can find a connected set $E \subset A$ such that $a, b \in E$. Then, we could decompose E into the union $E = (E \cap U_1) \cup (E \cap U_2)$ which would give that E is disconnected, against the choice of E. This contradiction completes the proof. □

Theorem 3.63. *The components of an open subset of $\mathbb{R}^n$ are open.*

Proof. If E is an open subset of $\mathbb{R}^n$, and $a \in E$, we can find $B(a, \epsilon) \subset E$ by definition of open set. Then, since $B(a, \epsilon)$ is connected (since it open and convex), it follows that $B(a, \epsilon) \subset C(a)$, and therefore the component $C(a)$ of E is open as well. □

We now consider the construction of complex functions from power series.

Definition 3.64. A series that is not convergent (in the sense discussed above) is said to be *divergent*. Let $\{x_n\}$ be a complex sequence. Suppose that for each $M > 0$ there exists $\nu \in \mathbb{N}$ such that for all $n > \nu$ it holds that $|x_n| > M$. Then we say that the sequence diverges to infinity.

Theorem 3.65. *A sequence in $\mathbb{C}$ converges if and only if it is a Cauchy sequence. Therefore, $\mathbb{C}$ is complete. Similar statements hold for $\mathbb{R}^n$ and $\mathbb{C}^n$.*

Proof. It is clear that any convergent sequence is Cauchy. We prove the converse. If $\{x_n\}$ is Cauchy, it is bounded. Therefore, $\{x_n\}$ has a convergent subsequence either by applying Theorem 3.35 if there are infinitely many values of x_n, or by observing that otherwise an element is repeated infinitely many times. But a Cauchy sequence that has a convergent subsequence is convergent, therefore completing the proof. □

Definition 3.66. We say that a series is absolutely convergent if $\sum_{n=0}^{\infty} |x_n|$ converges.

Exercise 3.67. Show that an absolutely convergent series converges.

Recall the following.

Definition 3.68. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Consider the limit $\lim_n \sup_{m\geq n} x_m$, which can be seen to exist, either finite or infinite.. This is the *limit superior* of $\{x_n\}$, and denoted $\limsup_n x_n$ or $\overline{\lim} x_n$. Similarly, one constructs the *limit inferior*, denoted as $\liminf_n x_n$ or $\underline{\lim} x_n$.

Let $\{x_n\}$ be a sequence. Then, we let S_n denote for each n the number $S_n = \sum_{i=1}^n x_i$. The sequence of values S_n associated to $\{x_n\}$ is a series. We say that a series is convergent, or divergent, based on whether the sequence S_n is convergent or divergent.

Theorem 3.69 (Cauchy Criterion)**.** *A series converges if and only if for every $\epsilon > 0$ there exists a $\nu \in \mathbb{N}$ such that $|x_n + \cdots + x_{n+p}| < \epsilon$ for each $n > \nu$ and $p > 0$.*

Proof. This follows directly from the fact that a sequence converges if and only if it is Cauchy. □

Let us now consider a sequence of complex functions $\{f_n\}$, all sharing the same domain, say $E \subset \mathbb{C}$. If for each value of $x \in E$ we have that the numerical sequence $\{f_n(x)\}$ is convergent, it follows that there is a "limit" function say f defined as $f(x) = \lim_n f_n(x)$. In this situation we say that the sequence *converges pointwise*. By definition of convergence, given a certain $\epsilon > 0$, for each $x \in E$ we can find $\nu \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \epsilon$ whenever $n > \nu$. However, notice that such ν can be dependent on x, in principle. We therefore pose the following definition.

Definition 3.70. Let $\{f_n\}$ be a sequence of complex functions as above. Then, we say that $\{f_n\}$ converges to a function f *uniformly*, if for each $\epsilon > 0$ we can find $\nu \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \epsilon$ whenever $n > \nu$ for all $x \in E$.

One of the reasons why the previous definition is useful is the following.

Theorem 3.71. *Let $\{f_n\}$ be a sequence of continuous functions defined on E. If $\{f_n\}$ converges uniformly to f, then f is continuous too.*

Proof. Let x^* be a point of E. Given $\epsilon > 0$, let ν be such that $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $n > \nu$ and all $x \in E$. For some chosen $n > \nu$, by continuity of f_n, we can find $\delta > 0$ such that $|f(x) - f(x^*)| < \frac{\epsilon}{3}$ whenever $|x - x^*| < \delta$. With such n, we then have

$$\begin{aligned} |f(x) - f(x^*)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x^*)| + |f_n(x^*) - f(x^*)| \\ &< \epsilon, \end{aligned}$$

which shows continuity of f at x^*. Since x^* was arbitrary, it follows that f is continuous. □

We have a Cauchy Criterion for the uniform convergence of sequences of functions as well.

Theorem 3.72 (Cauchy Criterion)**.** *The sequence $\{f_n\}$ converges uniformly on E if and only if for every $\epsilon > 0$ there exists $\nu \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m > \nu$ and for all $x \in E$.*

Proof. If $\{f_n\}$ converges uniformly to f, then let ν be chosen such that $|f(x) - f_n(x)| < \frac{\epsilon}{2}$ holds for all $n > \nu$ and for all x. Then, whenever $n, m > \nu$ we find that for all $x \in E$ one has

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Conversely, suppose that the condition in the statement holds true. Then, for each $x \in E$ the sequence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{C}$, and therefore converges. Therefore, for each x we have

$\lim_n f_n(x) = f(x)$ where we called $f(x)$ such limit value. We want to show that the sequence $\{f_n\}$ converges uniformly to f. For $\epsilon > 0$, let ν be such that $|f_n(x) - f_m(x)| < \epsilon/2$ for all $n, m > \nu$ and for all $x \in E$. Then, since $f_n(x) \longrightarrow f(x)$ for each $x \in E$, we have that $|f(x) - f_m(x)| \leq \epsilon/2 < \epsilon$, where we have let n go to infinity. Since the inequality holds for all $x \in E$, it follows that the convergence is uniform. □

The following result is very useful in practice too.

Proposition 3.73. *Suppose that $\{f_n\}$ is a sequence of functions such that $|f_n(x) - f_m(x)| \leq |a_n - a_m|$ for all $x \in E$, where $\{a_n\}$ is a convergent sequence in $\mathbb{C}$. Then the sequence $\{f_n\}$ is uniformly convergent.*

Proof. This is an exercise. □

We define series of functions in an analogous way as we have generalized sequences of numbers to functions. There is a variant of the previous result also for series.

Proposition 3.74 (M test)**.** *Let $\sum_n f_n$ be a series of functions such that $|f_n(x)| \leq Ma_n$ for all n sufficiently large, and all $x \in E$. If $\sum_n a_n$ is a convergent series, then if $\sum_n a_n$ converges, $\sum_n f_n$ converges uniformly.*

Proof. This is an exercise. □

We consider now a special type of series of functions. Namely, we consider the class of power series. These are series of type

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

where $a_n \in \mathbb{C}$ for each $n \in \mathbb{N}$, $z_0 \in \mathbb{C}$, and z is a complex variable. We will discuss the case where $z_0 = 0$ for notational simplicity, since everything can be adjusted to the more general case where z_0 is nonzero easily.

Definition 3.75. Let $\sum_{n=0}^{\infty} a_n z^n$ be a series. We define $\frac{1}{R} = \limsup_n \sqrt[n]{|a_n|}$, with the convention that $R = \infty$ if $\frac{1}{R} = 0$, and $R = 0$ if $\frac{1}{R} = \infty$. The value R is called the *radius of convergence* of the series.

Example 3.76. The geometric series, defined as $\sum_{n=0}^{\infty} z^n$ is convergent to $\frac{1}{1-z}$ for all z such that $|z| < 1$, and divergent whenever $|z| \geq 1$. This can be shown as for the case of the real geometric series, and we leave it to the reader to verify.

Theorem 3.77. *For a series $\sum_{n=0}^{\infty} a_n z^n$, the radius of convergence has the following properties:*

(i) *The series converges absolutely for every $z \in \mathbb{C}$ such that $|z| < R$. Moreover, of $0 \leq \rho < R$, the convergence is uniform for $|z| \leq \rho$.*
(ii) *If $|z| > R$, the series is divergent.*
(iii) *For $|z| < R$, the function defined by the series is a holomorphic function. The derivative is given by $\sum_{n=1}^{\infty} na_n z^{n-1}$, and it has the same radius of convergence R.*

Proof. (i) Let $|z| < R$, and let $\rho > 0$ be such that $|z| < \rho < R$. Then, it follows that $\frac{1}{\rho} > \frac{1}{R}$, and we can find ν such that $|a_n| < \frac{1}{\rho^n}$ for all $n > \nu$. Therefore, we find that $|a_n z^n| < (\frac{|z|}{\rho})^n$, which means that the series can be majorized by a convergent geometric series, and it therefore converges. To

prove that the convergence is uniform on a radius ρ ball with $\rho < R$, we can just choose ρ' such that $\rho < \rho' < R$ and proceed as before, showing that the M-test is applicable.

If $|z| > R$, we can choose ρ such that $R < \rho < |z|$. It follows that $\frac{1}{\rho} < \frac{1}{R}$, and therefore $|a_n| > (\frac{1}{\rho})^n$ for infinitely many n. Therefore, $|a_n z^n| > (\frac{|z|}{\rho^n})$ for infinitely many n, which implies that the series is not convergent because the terms are unbounded.

To show (iii), we first show that the series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ has the same radius of convergence R of the original series, and then show that it is its derivative. Since $\sqrt[n]{n} \longrightarrow 1$, it follows that $\limsup \sqrt[n]{na_n} = \limsup \sqrt[n]{a_n} = \frac{1}{R}$, and therefore the series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ has the same radius of convergence R. From (i), it follows that in the disc $B(0,R)$ the function $\sum_{n=1}^{\infty} n a_n z^{n-1}$ is defined as $\sum_{n=0}^{\infty} a_n z^n$. We want to show that the former is the derivative of the latter. We write, for $|z| < R$, $f(z) = \sum_{n=0}^{\infty} a_n z^n = S_n(z) + R_n(z)$, where we set $S_n(z) = \sum_{k=0}^{n} a_n z^n$, and $R_n(z) = \sum_{k=n+1} a_k z^k$. We also set $f_1(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \lim_n S_n'(z)$, where the derivative of $S_n(z)$ makes sense since for each n we have that $S_n(z)$ is a polynomial. Showing that $f'(z) = f_1(z)$ would complete the proof. Let z_0 be a point in the disc $B(0,\rho)$ with $\rho < R$, and let z be an arbitrary point in $B(0,\rho)$, different from z_0. Then, we have

$$\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0) \;=\; [\frac{S_n(z)-S_n(z_0)}{z-z_0} - S_n'(z_0)] + [S_n'(z_0) - f_1(z_0)] + \frac{R_n(z)-R_n(z_0)}{z-z_0}.$$

Since $R_n(z) - R_n(z_0) = \sum_{k=n+1}^{\infty} a_k (z^k - z_0^k)$, and $z^k - z_0^k = (z-z_0)\sum_{i=0}^{k-1} z^{k-i} z_0^i$, we find that

$$\frac{R_n(z)-R_n(z_0)}{z-z_0} \;=\; \sum_{k=n+1}^{\infty} a_k \sum_{i=0}^{k-1} z^{k-1-i} z_0^i,$$

from which it follows that

$$|\frac{R_n(z)-R_n(z_0)}{z-z_0}| \;\leq\; \sum_{k=n+1}^{\infty} k|a_k|\rho^{k-1}.$$

Since the series $\sum_{n=1}^{\infty} n|a_n|\rho^{n-1}$ is convergent, it follows that the remainder $\sum_{k=n+1}^{\infty} k|a_k|\rho^{k-1}$ tends to zero as n goes to $+\infty$. Therefore, we can find ν_1 such that $n > \nu_1$ implies that

$$|\frac{R_n(z)-R_n(z_0)}{z-z_0}| \;<\; \frac{\epsilon}{3}.$$

Moreover, the fact that $\lim_n S_n'(z_0) = f_1(z_0)$, implies that for $n > \nu_2$, for some ν_2, we have also

$$|S_n'(z_0) - f_1(z_0)| \;<\; \frac{\epsilon}{3}.$$

Therefore, for any fixed $n > \max\{\nu_1, \nu_2\}$, both above inequalities hold. Since S_n' is the derivative of the partial sum S_n, we find some $\delta > 0$ such that $|z - z_0| < \delta$ implies

$$|\frac{S_n(z)-S_n(z_0)}{z-z_0} - S_n'(z_0)| \;<\; \frac{\epsilon}{3}.$$

Combining all inequalities, we have that

$$|\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0)| \;<\; \epsilon,$$

for all z such that $|z - z_0| < \delta$. This means that f_1 is the derivative of f, which completes the proof. □

Corollary 3.78. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a series with radius of convergence R. Then, f is differentiable infinitely many times in $B(0,R)$.*

Proof. We can iterate the procedure of Theorem 3.77. □

Theorem 3.79 (Abel's Limit Theorem)**.** *Suppose that* $\sum_{n=0}^{\infty} a_n = A$*, where the series* $\sum_{n=0}^{\infty} a_n z^n$ *has radius of convergence* $R = 1$*. Then, setting* $f(z) = \sum_{n=0}^{\infty} a_n z^n$*, we have that* $\lim_{z\to 1} f(z) = A$*, as* z *approaches* 1 *in such a way that* $\frac{|1-z|}{1-|z|}$ *remains bounded.*

Proof. Without loss of generality, we can assume that $A = 0$, since upon changing a_0 this can be achieved. Write $s_n = \sum_{k=0}^{n} a_k$, set $s_n(z) = \sum_{k=0}^{n} a_k z^k$ and rewrite the terms of $s_n(z)$ as

$$\begin{aligned} s_n(z) &= a_0 + a_1 z + \cdots + a_n z^n \\ &= s_0 + (s_1 - s_0)z + \cdots + (s_n - s_{n-1})z^n \\ &= s_0(1-z) + s_1(z - z^2) + \cdots + s_{n-1}(z^{n-1} - z^n) + s_n z^n \\ &= (1-z)(s_0 + s_1 z + \cdots + s_{n-1} z^{n-1}) + s_n z^n. \end{aligned}$$

Since $s_n z^n \longrightarrow 0$, it follows that

$$f(z) = (1-z)\sum_{n=0}^{\infty} s_n z^n.$$

For z such that $|1 - z| \leq K(1 - |z|)$, for given $\epsilon > 0$ we can find ν such that for $n > \nu$ we have $|s_n| < \epsilon$. Then, $\sum s_n z^n$ can be bounded starting the sum from $\nu + 1$ on as

$$\begin{aligned} |\sum_{n=\nu+1}^{\infty} s_n z^n| &\leq \sum_{n=\nu+1}^{\infty} |s_n z^n| \\ &\leq \epsilon \sum_{n=\nu+1}^{\infty} |z^n| \\ &= \epsilon |z|^{\nu+1} \frac{1}{1-|z|} \\ &< \frac{\epsilon}{1-|z|}, \end{aligned}$$

where we have used the limit of the geometric series. Then, we find that

$$|f(z)| \leq |1-z||\sum_{k=0}^{\nu} s_k z^k| + K\epsilon,$$

where the first terms in the right-hand-side can be made arbitrarily small upon choosing z close enough to 1. This shows that $f(z) \xrightarrow{z\to 1} 0$ as required. □

We now consider the problem of constructing some well known complex functions by means of power series. We consider first the epxonential function. This arises as the solution of the differential equation $f'(z) = f(z)$, with initial value $f(0) = 1$. If we let $f(z)$ be defined as a power series, i.e. if we set $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we can compute its derivative (inside its radius of convergence) as $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. Equating the two power series, we obtain an infinite system of equations for the coefficients a_n given by $a_{n-1} = n a_n$, where $a_0 = 1$ by the initial value. By induction, we can show that $a_n = \frac{1}{n!}$. We therefore obtain the definition of the exponential function as a power series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

This makes sense within the radius of convergence of the power series. Since we have that $\sqrt[n]{n!} \longrightarrow +\infty$, it follows that $R = +\infty$. Therefore, as shown in Theorem 3.77, it follows that the series converges on the whole complex plane.

It is possible to show, by means of the defining differential equation and the Leibniz rule, that the usual addition identity for the exponential function holds, i.e. that $e^{a+b} = e^a e^b$. In fact, one has that $D(e^z \cdot e^{c-z}) = 0$, which implies that $e^z \cdot e^{c-z}$ is a constant whose value is determined by setting $z = 0$ in the power series. Hence, we get $e^z \cdot e^{c-z} = e^c$. Taking $z = a$ and $c = a + b$ one obtains the claimed equality. As a consequence, we find that $e^z \cdot e^{-z} = 1$, which implies that e^z is never zero.

Since we have already introduced the complex exponent at the beginning of the course via trigonometric forms, the natural question of whether the two definitions are equivlant arises. To see that, we consider the power series definition of trigonometric functions. We define

$$\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2}, \\ \sin z &= \frac{e^{iz} - e^{-iz}}{2}. \end{aligned}$$

Using the power expansion definition of e^{iz}, we obtain

$$\begin{aligned} \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots, \\ \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots. \end{aligned}$$

So, for real z, one sees that these are the usual real Taylor expansions for the cosine and sine functions, showing that the current definition is consistent with the geometric one previously used. In particular, one finds again Euler's formula as

$$e^{iz} = \cos z + i \sin z.$$

Moreover, one finds the well known identities

$$\begin{aligned} \cos^2 z + \sin^2 z &= 1, \\ D(\cos z) &= -\sin z, \\ D(\sin z) &= \cos z, \\ \cos(z + w) &= \cos z \cos w - \sin z \sin w \\ \sin(z + w) &= \cos z \sin w + \sin z \cos w. \end{aligned}$$

One can also define the other trigonometric functions from cosine and sine just obtained above, by the usual formulas.

Next, we consider the complex logarithm. By definition, $z = \log w$ is a root of the equation $e^z = w$. As we have shown before, $e^z \neq 0$ for all $z \in \mathbb{C}$, and it follows that there is no logarithm of 0. Next, observe that the equation $e^{x+iy} = w$, using our definitions of exponential function, gives the two equations

$$\begin{aligned} e^x &= |w|, \\ e^{iy} &= \frac{w}{|w|}. \end{aligned}$$

The first equation is just the equation for the real logarithm, which provides a unique solution as $x = \log |w|$. Here we notice that since $w \neq 0$, it follows that automatically $|w| > 0$, and therefore

the logarithm of $|w|$ is well defined. In the second equation, we have that $\frac{w}{|w|}$ has unitary modulus, and therefore e^{iy} lies in the unit circle. It follows that there exists a unique angle $\theta \in [0, 2\pi)$ such that $e^{i\theta} = \frac{w}{|w|}$. Such θ is a solution to the equation. However, by periodicity of the cosine and sine functions, we know that there will be infinitely many solutions to the equation obtained by adding multiples of 2π, i.e. any value $\theta + 2k\pi$ is a solution of the equation that defines the complex logarithm, where $k \in \mathbb{Z}$. Combining the two solutions we find that

$$\log w \quad = \quad \log |w| + i \arg w,$$

where we have called the *argument* of w, written $\arg w$, any angle $\theta + 2k\pi$ where $k \in \mathbb{Z}$. This is in line with our definitions of geometric form of a complex number.

Another convention that is usually employed, is that a^b, where $a, b \in \mathbb{C}$ with $a \neq 0$, means that complex number $\exp(b \log a)$. If a is not real, a^b will have infinitely many values.

As a consequence of the addition property proved for the exponential function we find that

$$\begin{aligned} \log(z_1 z_2) &= \log z_1 + \log z_2, \\ \arg(z_1 z_2) &= \arg z_1 + \arg z_2, \end{aligned}$$

in the sense of equality of (infinite) sets.

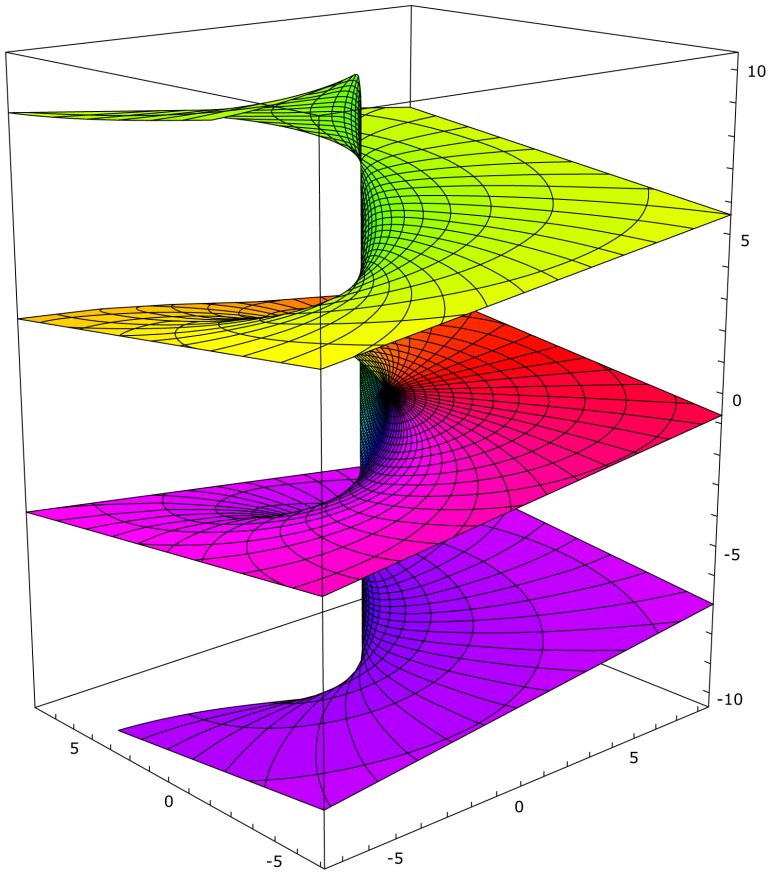


FIGURE 2. Multi-valued logarithm showing a surface branched over zero (from Wikipedia)

A similar argumentation can be given for the inverse of the cosine function. This is defined by solving the equation $\cos z := \frac{1}{2}(e^{iz} + e^{-iz}) = w$. Solving the quadratic equation for e^{iz} we obtain

$$e^{iz} \quad = \quad w \pm \sqrt{w^2 - 1},$$

which in turn gives

$$\arccos w \quad = \quad -i \log\Big(w \pm \sqrt{w^2 - 1}\Big).$$

This has infinitely many values, since so does the complex logarithm. The inverse sine is simply given by

$$\arcsin w \quad = \quad \frac{\pi}{2} - \arccos w.$$

We complete this section by considering *conformal mappings*. Let $z(t)$ denote a function that defines an arc γ in $\mathbb{C}$. So, $z(t)$ is a parametrization that takes values for t in $[\alpha, \beta] \subset \mathbb{R}$. Suppose that Ω is a region, and $z(t) \in \Omega$ for each $t \in [\alpha, \beta]$. Suppose that f is a complex function defined (and continuous) in Ω. Then, we can define a new parametrized arc $w(t) = f(z(t))$ by considering the outcome of applying f to each value $z(t)$. If $z'(t)$ exists, in the sense of a map from the reals to $\mathbb{R}^2$, and f is holomorphic, it follows that $w'(t)$ exists as a map from $[\alpha, \beta]$ to $\mathbb{R}^2$. Moreover, one can check that this is just given by

$$w'(t) \quad = \quad f'(z(t))z'(t).$$

Suppose that $t_0 \in [\alpha, \beta]$, and consider $z_0 = z(t_0)$, such that $z'(t_0), f'(z_0) \neq 0$. Clearly, one gets that $w'(t_0) \neq 0$. The curve $\gamma' = f(z(t))$ has a tangent at $w_0 = f(z_0)$. The direction of the tangent is determined by the argument of $w'(t_0)$, where

$$\arg w'(t_0) \quad = \quad \arg f'(z_0) + \arg z'(t_0).$$

In other words, $\arg f'(z_0)$ is determined by the difference of the directions between γ' and γ (at the specified point), and therefore this difference is independent of the curve γ. So, the angle between the tangents to γ and γ' at the given point depends only on f. This means that curves passing through z_0 which are tangent to each others are mapped onto curves that have a common tangent at the corresponding point w_0. Similarly, curves that form a certain angle at z_0 are mapped to curves that have the same angle at w_0. Therefore, a holomorphic map f is said to be conformal at points where $f'(z) \neq 0$.

Considering the modulus, the equation

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} \quad = \quad |f'(z_0)|,$$

geometrically means that a line segment passing through z_0 is contracted or expanded (in the limit) by a $|f'(z_0)|$ factor. So, the linear change of scale at z_0 is not dependent on the direction at a given point z_0 of conformality for f.

Moreover, both types of conformality (in argument and modulus) together imply that the derivative $f'(z_0)$ exists.

Let us now assume that both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous. Then, we can write $w'(t_0) = \frac{\partial f}{\partial x}x'(t_0) + \frac{\partial f}{\partial y}y'(t_0)$, with partial derivatives evaluated at z_0. Therefore we have

$$w'(t_0) \quad = \quad \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})z'(t_0) + \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})\overline{z'(t_0)}.$$

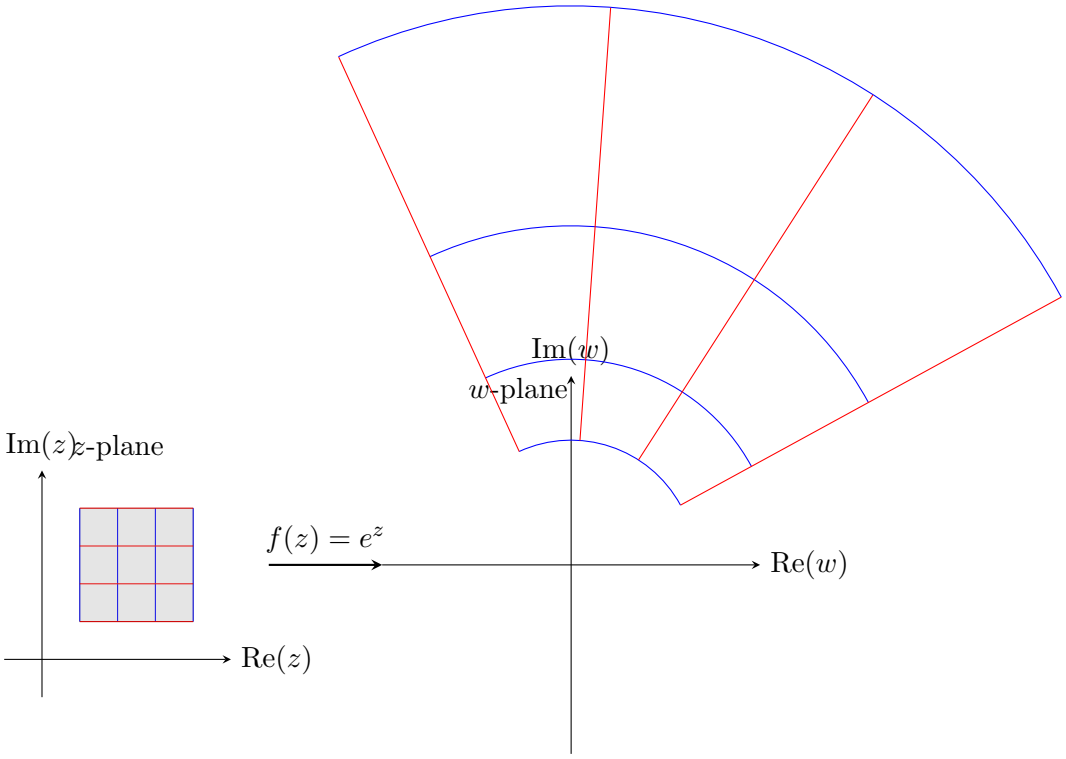


FIGURE 3. Exponential map conformal transformation

When Angles are preserved, as discussed above, the argument $\arg(\frac{w'(t_0)}{z'(t_0)})$ is independent of $\arg z'(t_0)$. Therefore, we have that the quantity

$$\frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}) + \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})\frac{\overline{z'(t_0)}}{z'(t_0)}$$

has constant argument. As we vary $\arg z'(t_0)$, the point $\frac{w'(t_0)}{z'(t_0)}$ lies in a circle of radius $\frac{1}{2}|\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}|$. If the argument of $\frac{w'(t_0)}{z'(t_0)}$ is constant, it means that the corresponding radius must vanish. And therefore we find that $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$. This latter equation corresponds to the Cauchy-Riemann equations, once we set $u(x, y) = \operatorname{Re} f$ and $v(x, y) = \operatorname{Im} f$. Therefore, it follows that f is holomorphic.

A similar argument might be employed considering that the quantity $\frac{w'(t_0)}{z'(t_0)}$ has change of scale equal in all directions. Leading to an equation of type $\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} = 0$, which shows that $\bar{f}$ is holomorphic.

We say that a function is *indirectly conformal* if it is the conjugate of an analytic function with nonvanishing derivative. This name follows from the fact that it preserves the sizes, but it reverses the orientation of the angles.

The fact that copnformal maps rescale the neighborhood of a point by a factor of $|f'(z)|$, we expect that the areas are rescaled by a factor of $|f'(z)|^2$. This is indeed what happens. The length

of an arc γ defined as $z(t) = x(t) + iy(t)$, where $t \in [a, b]$, is given by

$$\begin{aligned} L(\gamma) &= \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt \\ &= \int_a^b |z'(t)|dt. \end{aligned}$$

If $\sigma(t)$ is the image curve $w(t) = f(z(t))$, where f is conformal, we have that the new length is

$$L(\sigma) = \int_a^b |f'(z)||z'(t)|dt,$$

where we have used the fact that $w'(t) = f'(z(t))z'(t)$. So, the local rescaling $|f'(z(t))|$ modifies the length, and the integration is due to the fact that the rescaling is local (we had it in a limit process). Let us now consider areas, and let E be a set whose area is given by

$$A(E) = \int\int_E dxdy.$$

This area can be evaluated via a double Riemann integral. If $f(z) = u(x, y) + iv(x, y)$, is bijective and diffefrentiable, the are of the image $E' = f(E)$ is given by

$$A(E') = \int\int_E |\det J(u, v)|dxdy,$$

where $J(u, v)$ is the Jacobian with respect to the functions u and v. If f is conformal on an open set U containing E, the Cauchy-Riemann equations allow us to rewrite $\det J(u, v) = |f'(z)|^2$, and therefore one has the formula

$$A(E') = \int\int_E |f'(z)|^2 dxdy,$$

which shows that the areas are rescaled by a $|f'(z)|^2$ factor. Once again, this factor is local, and therefore it needs to be integrated over the whole set.

An important class of conformal maps is the class of *linear fractional transformations*. These are maps defined by the assignment

$$z \mapsto \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$. These are also called Möbius transformations in the case of complex numbers as in our case, but this definition can be greatly extended. We will not concern ourselves with the general case, though.

These transformations take a particularly interesting meaning in the Riemann sphere, which is an extension of the complex numbers as follows.

We introduce a point denoted by ∞, and construct a mapping that gives a bijection between the sphere (regular sphere) minus the north pole N, and the complex numbers $\mathbb{C}$. The mapping is defined via *stereographic projection* from the North Pole $N = (0, 0, 1)$. For any point P on the sphere (excluding N), a unique line passes through N and P, intersecting the complex plane (embedded as the equatorial plane or tangent at the South Pole) at a point z. As $P \to N$, the magnitude of the corresponding complex number $|z| \to \infty$. Thus, the North Pole N is identified with the point at infinity, ∞.

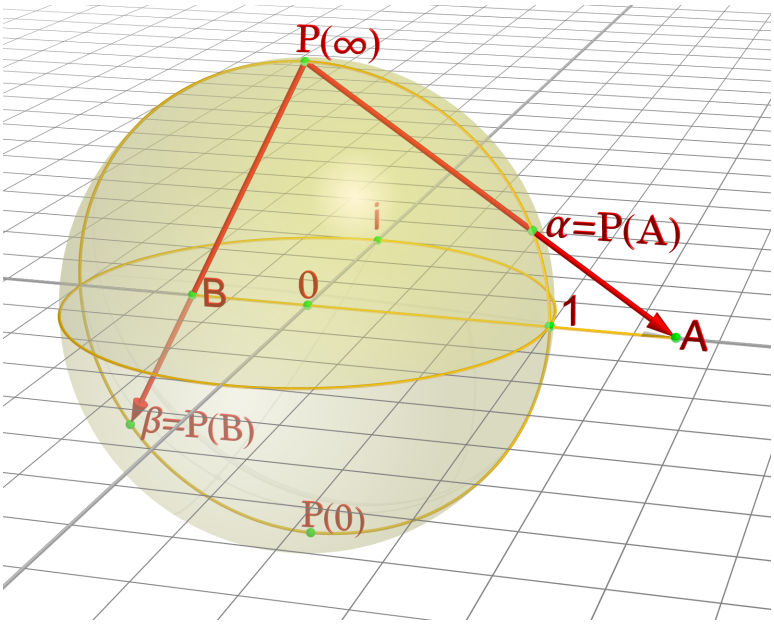


FIGURE 4. Riemann sphere and corresponding stereographic projection (from Wikipedia)

Given a point (x_1, x_2, x_3) on the sphere $S^2 \setminus \{N\}$, the corresponding complex number $z = x + iy$ is given by the projection from the North Pole $N = (0, 0, 1)$:

$$z = \frac{x_1 + ix_2}{1 - x_3}.$$

Separating the real and imaginary parts, we have:

$$\begin{aligned} \mathrm{Re}(z) &= \frac{x_1}{1 - x_3}, \\ \mathrm{Im}(z) &= \frac{x_2}{1 - x_3}. \end{aligned}$$

To map a complex number $z \in \mathbb{C}$ back to the sphere, we use the inverse transformation:

$$\begin{aligned} x_1 &= \frac{2\mathrm{Re}(z)}{|z|^2 + 1}, \\ x_2 &= \frac{2\mathrm{Im}(z)}{|z|^2 + 1}, \\ x_3 &= \frac{|z|^2 - 1}{|z|^2 + 1}. \end{aligned}$$

This construction gives a notion of extended Complex numbers, similarly to how adding $\pm\infty$ extends the real numbers. The notation for the extended complex numbers is usually $\hat{\mathbb{C}}$, or $\mathbb{C}_\infty$. The set $\mathbb{C}_\infty$ can be endowed with a topology inherited from the sphere. Here the neighborhoods of ∞ are the complements of closed discs in $\mathbb{C}$.

The Möbius transformations can be extended to maps defined on the Riemann sphere, by setting $\infty \mapsto \frac{a}{c}$, and $-\frac{d}{c} \mapsto \infty$. If S is a Möbius transformation, with $a, b, c, d \in \mathbb{C}$ as above, its inverse is given by

$$S^{-1}(w) = \frac{dw - b}{-cw + a}.$$

It follows that S is a homeomorphism of $\mathbb{C}_\infty$ onto $\mathbb{C}_\infty$. The global conformal group of the Riemann sphere is the finite dimensional group of Möbius transformations, which is often denoted by $PSL_2(\mathbb{C})$.

If the coefficients of S satisfy $ad - bc = 1$, we say that the transformation is normalized. We can describe the Möbius transforamtions conveniently by means of *homogeneous coordinates*. Writing $z = \frac{z_1}{z_2}$ and $w = \frac{w_1}{w_2}$, then we can write $S(z) = w$ using the matrix notation

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad = \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

With this notation, one can see that composition of Möbius transformations just becomes the usual product of the corresponding matrices, so that $S_1 S_2$ is determined by the matrix

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix},$$

where $S_i(z) = \frac{a_i z + b_i}{c_i z + d_i}$ for $i = 1, 2$. It follows that the Möbius tranasformations form a group, with the identity matrix representing the identity element.

One reason conformal maps are important in physics is that they appear naturally in *conformal field theory* (CFT), especially in two dimensions. Very roughly, a conformal field theory is a theory whose basic quantities behave well under angle-preserving changes of coordinates.

This is one of the reasons complex analysis plays such an important role in physics: in two real dimensions, conformal maps are closely related to holomorphic functions as we have seen above. Thus the same functions we study in complex variables also describe symmetries of certain physical systems.

In two-dimensional CFT, one studies *fields*, which are functions or observables defined on the plane. A particularly important class is that of *primary fields*. If $w = f(z)$ is a conformal change of coordinates, then a primary field of conformal weight h transforms according to

$$\widetilde{\phi}(w) \quad = \quad \big(f'(z)\big)^{-h} \phi(z).$$

So the field does not remain literally unchanged: it picks up a factor involving the derivative of the conformal map. This reflects the fact that the field has a certain scaling behavior.

A basic quantity in CFT is the *two-point correlation function*. For a holomorphic primary field on the complex plane, it has the form

$$\langle \phi(z_1) \phi(z_2) \rangle \quad = \quad \frac{C}{(z_1 - z_2)^{2h}},$$

where C is a constant and $\langle \cdot \rangle$ denotes the correlation function.

Now consider a Möbius transformation

$$w \quad = \quad f(z) = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0.$$

Its derivative is

$$f'(z) \quad = \quad \frac{ad - bc}{(cz + d)^2}.$$

Also,

$$w_1 - w_2 \quad = \quad \frac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)}.$$

Using the transformation rule for the field, we get

$$\langle\widetilde{\phi}(w_1)\widetilde{\phi}(w_2)\rangle \;=\; \big(f'(z_1)\big)^{-h}\big(f'(z_2)\big)^{-h}\langle\phi(z_1)\phi(z_2)\rangle.$$

Substituting the formulas above gives

$$\langle\widetilde{\phi}(w_1)\widetilde{\phi}(w_2)\rangle \;=\; \frac{C}{(w_1-w_2)^{2h}}.$$

So the correlation function has the *same form* in the new coordinate system.

This is a simple illustration of conformal invariance: although the coordinates change, the physical quantity transforms in a very controlled way. In two dimensions this becomes especially powerful, because conformal maps are governed by complex analytic functions. That is one of the main reasons why complex variables is so useful in modern theoretical physics.

4. Line integrals and Cauchy theorem

To completely settle the problem of showing that the derivative of a holomorphic function is continuous, we will make use of complex integration. This will give us some important tools associated to Cauchy's theorem. It is possible to show this result, along the fact that all derivatives of holomorphic functions exist, without the use of integration. However, these results require tools from topology which are not available to us in this course.

Let $f:[a,b]\longrightarrow\mathbb{C}$ be a complex-valued function defined over a real interval, and such that the real and imaginary parts are real integrable over $[a,b]$. We can set by definition:

$$\int_a^b f(t)dt \;:=\; \int_a^b u(t)dt + i\int_a^b v(t)dt, \tag{4}$$

where u and v are real and imaginary parts of f, respectively. Each integral in the right-hand-side of (4) is real, therefore we can "recycle" most of the properties of the real integral to extend them to the current setup. For example, linearity of the complex integral in (4) follows from linearity of the real integral. In fact, we have that

$$\int_a^b \alpha f(t)dt \;=\; \alpha\int_a^b f(t)dt, \tag{5}$$

$$\int_a^b (f(t)+g(t))dt \;=\; \int_a^b f(t)dt + \int_a^b g(t)dt \tag{6}$$

for any $\alpha\in\mathbb{C}$, and any f and g as above.

Exercise 4.1. Prove the previous statement.

In addition, we have that

$$\Big|\int_a^b f(t)dt\Big| \;\leq\; \int_a^b |f(t)|dt,$$

for $a\leq b$. We prove this as follows. First, observe that if $\int_a^b f(t)dt = 0$, the inequality holds automatically. We can therefore assume that $\int_a^b f(t)dt\neq 0$. Then, let $\theta = \arg\int_a^b f(t)dt$. Applying

(5) with $\alpha = e^{-i\theta}$, we have

$$\begin{aligned} |\int_a^b f(t)dt| &= \mathrm{Re}(e^{-i\theta}\int_a^b f(t)dt) \\ &= \mathrm{Re}(\int_a^b e^{-i\theta}f(t)dt) \\ &= \int_a^b \mathrm{Re}(e^{-i\theta}f(t))dt \\ &\leq \int_a^b |f(t)|dt. \end{aligned}$$

Let γ indicate a piecewise differentiable arc in $\mathbb{C}$, with some parametrization $z(t)$ for $t \in [a,b]$. Let f be a complex function which is continuous on γ. Then, it follows that the composite $f(z(t))$ is a continuous function $[a,b] \longrightarrow \mathbb{C}$. We can therefore use the definition in (4) to set

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt, \tag{7}$$

which we call the *complex line integral*. When z' is not continuous on the whole interval $[a,b]$, since γ is piecewise differentiable, we can subdivide the interval in finitely many pieces such that z' is continuous over them. We can therefore define the integral in (7) as a finite sum over these intervals. Since the arc γ could potentially be expressed by a change of parameter, i.e. by means of a piecewise differentiable increasing function $t(\tau)$ which maps $[\alpha,\beta]$ onto $[a,b]$, a natural question arises: Would the integral as defined above change? We see now that this is not the case.

In fact, if we set $\hat{z}(\tau) := z(t(\tau))$, we have that

$$\begin{aligned} \int_\gamma f(z)dz &= \int_a^b f(z(t))z'(t)dt \\ &= \int_\alpha^\beta f(z(t(\tau)))z'(t(\tau))t'(\tau)d\tau \\ &= \int_\alpha^\beta f(z(t(\tau)))\frac{d}{d\tau}(z(t(\tau)))d\tau \\ &= \int_\alpha^\beta f(\hat{z}(t))\hat{z}'(\tau)d\tau \\ &= \int_\gamma f(\hat{z})d\hat{z}, \end{aligned}$$

by applying the change of integration variable to the real and imaginary parts of fz', and considering that $z'(t(\tau))t'(\tau) = \frac{d}{d\tau}(z(t(\tau)))$.

Given γ, we define the opposite arc $-\gamma$ as $z(-t)$, where $t \in [-b,-a]$. Then, one has

$$\begin{aligned} \int_{-\gamma} f(z)dz &= \int_{-b}^{-a} f(z(-t))(-z'(-t))dt \\ &= -\int_a^b f(z(t))z'(t)dt \\ &= -\int_\gamma f(z)dz. \end{aligned}$$

We now define an operation of addition of arcs in the following sense. If γ is subdivided into subarcs $\gamma_1, \ldots, \gamma_n$, then we write $\gamma = \gamma_1 + \cdots + \gamma_n$ and the integral defined above has the property that

$$\int_\gamma f(z)dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)dz.$$

We define the integral with respect to $\bar{z}$ by the formula

$$\int_\gamma f\overline{dz} = \overline{\int_\gamma \bar{f}dz}, \tag{8}$$

where the integral on the right-hand-side uses the previous definition. Moreovere, one defines the integrals

$$\begin{aligned} \int_\gamma f dx &= \frac{1}{2}(\int_\gamma f dz + \int_\gamma f\overline{dz}) \\ \int_\gamma f dy &= \frac{1}{2i}(\int_\gamma f dz - \int_\gamma f\overline{dz}), \end{aligned}$$

where x and y are real and imaginary parts of the complex number $z = x + iy$.

We consider now the integral with respect to *arc length*. This is defined as

$$\begin{aligned} \int_\gamma f ds &= \int_\gamma f|dz| \\ &= \int_\gamma f(z(t))|z'(t)|dt. \end{aligned}$$

The change of variable formula now gives the result

$$\int_{-\gamma} f|dz| = \int_\gamma f|dz|.$$

We still have

$$\int_\gamma f(z)|dz| = \sum_{i=1}^{n} \int_{\gamma_i} f(z)|dz|.$$

Moreover, one finds the inequality

$$|\int_\gamma f dz| \leq \int_\gamma |f| \cdot |dz|.$$

Exercise 4.2. Prove the previous inequality.

The arc length integral of the function $f(z) = 1$ is by definition the arc length of the curve γ.

Example 4.3. We show that the arc length of the circle is equal to the perimeter of the circle, as one would expect. In fact, a parametrization of the circle of radius ρ centered at the point $a \in \mathbb{C}$,

which we indicate by $\partial D(a,\rho)$, is given by $z(t) = a + \rho e^{it}$, where $t \in [0, 2\pi]$. Then, one has

$$\begin{aligned}\int_{\partial D(a,\rho)} |dz| &= \int_0^{2\pi} |z'(t)|dt \\ &= \int_0^{2\pi} |\rho i e^{it}|dt \\ &= \rho \int_0^{2\pi} dt \\ &= 2\pi\rho.\end{aligned}$$

We consider now the following question: When does the arc integral depend only on the end points of the arc γ? Or better to say: Is there a class of functions whose values only depend on the end points of γ?

Proposition 4.4. *For a given f, the integral $\int_\gamma f dz$ depends only on the end points of γ if and only if $\int_\Gamma f dz = 0$ for every loop Γ.*

Proof. Suppose that f is such that $\int_\gamma f dz$ depends only on the end points of γ. Then, let Γ be a loop based at $\alpha \in \mathbb{C}$, i.e. we have a parametrization $z(t)$, $t \in [a, b]$, such that $z(a) = z(b) = \alpha$. If Γ is constant, the derivative of z is zero, and the integral is zero. We can therefore assume that Γ is not constantly equal to α, i.e. it is not the trivial loop at α. Then we let β be any point on Γ different from α, and let $z(t^*) = \beta$. Let γ_1 denote the path from α to β given by $z(t)$, $t \in [a, t^*]$. Similarly, we let γ_2 indicate the path from β to α given by $z(t)$ with $t \in [t^*, b]$. Therefore we have that

$$\begin{aligned}\int_\Gamma f dz &= \int_{\gamma_1} f dz + \int_{\gamma_2} f dz \\ &= \int_{\gamma_1} f dz - \int_{-\gamma_2} f dz \\ &= 0,\end{aligned}$$

where the last equality is due to the fact that both γ_1 and $-\gamma_2$ are arcs connecting α to β, and therefore the integrals of f on these paths are the same by assumption.

Conversely, suppose that f is such that $\int_\Gamma f dz = 0$ for every loop Γ. Let γ_1 and γ_2 be arcs from α to β. By reversing the orientation of one of the two paths, say γ_2, we obtain a closed loop $\gamma_1 - \gamma_2$. Then, we have

$$\begin{aligned}\int_{\gamma_1} f dz - \int_{\gamma_2} f dz &= \int_{\gamma_1} f dz + \int_{-\gamma_2} f dz \\ &= \int_\Gamma f dz \\ &= 0,\end{aligned}$$

from which we find $\int_{\gamma_1} f dz = \int_{\gamma_2} f dz$ as required. □

The next result provides a necessary and sufficient condition for a function f to have the property above. Before stating and proving the result, we observe that using the definitions we can write the identity

$$\int_\gamma f dz = \int_\gamma (u dx - v dy) + i \int_\gamma (u dy + v dx),$$

where we use $f(x,y) = u(x,y)+iv(x,y)$, with $z = x+iy$. Therefore, one can define the arc integrals by means of integrals of type $\int_\gamma (pdx + qdy)$ equivalently. This allows us to give the theorem in a convenient form.

Theorem 4.5. *The integral $\int_\gamma (pdx+qdy)$, for continuous p and q defined on the region Ω, depends only on the end points of γ if and only if there exists a function $U(x,y)$ in Ω with partial derivatives satisfying the equations $\frac{\partial U}{\partial x} = p$ and $\frac{\partial U}{\partial y} = q$.*

Proof. First, suppose that a function satisfying such equations for its partial derivatives exists. Then, we have

$$\begin{aligned}\int_\gamma (pdx + qdy) &= \int_a^b (\frac{\partial U}{\partial x}x'(t) + \frac{\partial U}{\partial y}y'(t))dt \\ &= \int_a^b \frac{d}{dt}U(x(t),y(t))dt \\ &= U(x(b),y(b)) - U(x(a),y(a)),\end{aligned}$$

where we have used the notion of total derivative of U with respect to t through its functions $x(t)$ and $y(t)$. Since the last term does not depend on γ, but only on $x(a), x(b), y(a)$ and $y(b)$, which are determined by the end points of $z(t)$, parametrization of γ, the first part of the proof is complete.

Conversely, assume that the integral does not depend on the path γ joining two end points, but only on the end points. We want to define a function U on Ω in such a way that its partial derivatives give p and q. To this purpose, let us choose a point $z_0 = (x_0, y_0) \in \Omega$. Recall that by Theorem 3.60, any other point $z = (x,y) \in \Omega$ can be joined to (x_0,y_0) by a polygonal curve, in other words, there always exists a polygonal curve γ joining (x_0,y_0) to another chosen point of Ω. For a given γ joining (x_0,y_0) to (x,y), we define

$$U(x,y) = \int_\gamma (pdx + qdy).$$

We need to argue that the function is well defined, but this follows immediately from the fact that by hypothesis we have that $\int_\gamma (pdx+qdy)$ does not depend on γ, but only on the end points (x_0,y_0) and (x,y). Therefore, the assignment for $U(x,y)$ is unambiguous, and the function is well defined. We want to show that the partial derivatives of U give p and q as claimed. When we join (x_0,y_0) to (x_1,y_1), we will have one final line segment in a disc neighborhood $D(z,\rho) \subset \Omega$, since Ω is open. By possibly making a subdivision of this segment, we can get a polygonal curve γ such that the last arc is parallel to the real axis of $\mathbb{C}$, and therefore such that $y = y_1$ is constant. Therefore, if $\gamma = \gamma_1 + \cdots + \gamma_n$, where γ_n is horizontal—i.e. with y constant along γ_n—it follows that

$$\begin{aligned}U(x_1,y_1) &= \sum_{i=1}^{n-1}\int_{\gamma_i}(pdx+qdy) + \int_{\gamma_n} pdx \\ &= \sum_{i=1}^{n-1}\int_{\gamma_i}(pdx+qdy) + \int_{x^*}^{x_1} pdx\end{aligned}$$

where $\int_{\gamma_n} qdy = 0$ since γ does not depend on y because γ_n is horizontal, and x^* is the x-value of the initial point (x^*,y^*) of γ_n. A similar expression allows us to compute U on horizontal lines around (x_1,y_1). Computing the partial derivative of U at the point $z_1 = (x_1,y_1)$, the sums $\sum_{i=1}^{n-1}\int_{\gamma_i}(pdx+qdy)$ are irrelevant since to compute the derivative we only need values around

(x, y) along horizontal lines. It therefore follows that

$$\frac{\partial U}{\partial x}|_{(x_1,y_1)} = p(x_1, y_1).$$

Similarly, constructing a polygonal curve γ whose last arc is vertical, i.e. it has constant value for x, we obtain that

$$\frac{\partial U}{\partial y}|_{(x_1,y_1)} = q(x_1, y_1).$$

This completes the proof, since $(x_1, y_1) \in \Omega$ was arbitrary. □

One usually writes $dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$, meaning that if p and q are the partial derivatives of a function U as in the theorem, then $pdx + qdy$ is an exact differential. We notice that such a U is determined only up to constant, since if $\frac{\partial U}{\partial x} = p$ and $\frac{\partial U}{\partial y} = q$, also $U + k$ satisfies the same equations for ant constant k.

Now, suppose that $f(z)dx + if(z)dy$ is an exact differential for some continuous f. Then, let F be such that $dF = f(z)dx + if(z)dy$. This means that

$$\begin{aligned} \frac{\partial F}{\partial x} &= f(z) \\ \frac{\partial F}{\partial y} &= if(z), \end{aligned}$$

which implies the Cauchy-Riemann equation $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$. Therefore, F is holomorphic.

We therefore have the following.

Corollary 4.6. *The integral $\int_\gamma f dz$, for a continuous f defined on a region Ω, depends only on the end points of γ if and only if f is the derivative of an holomorphic function.*

Example 4.7. Applying the results above, one finds that $\int_\gamma (z-a)^n dz = 0$ for any loop γ whenever $n \geq 0$, since $f(z) = (z-a)^n$ is the derivative of $F(z) = \frac{(z-a)^{n+1}}{n+1}$. When n is negative and $n \neq -1$, if γ is a loop such that a is not on γ, we would again find that the integral is zero. However, for $n = -1$, a direct computation shows that the integral on the circle of radius ρ centered at a gives $\int_\gamma f(z)dz = 2\pi i$. This anomaly is related to the fact that we are not able to define a logarithm with a single branch, due to the multiplicity of possible choices we have encountered when defining log. If we consider the integral over a γ that does not contain the point a inside the area it delimits, then the integral would be zero.

Our main purpose now is to obtain Cauchy's theorem. We will proceed in several stages, first of which is the case over a rectangle. We let R denote a rectangle in $\mathbb{C}$, i.e. the set of points $(x, y) \in \mathbb{C}$ such that $x \in [a, b]$ and $y \in [c, d]$. The perimeter ∂R of R is oriented in anticlockwise manner, and we therefore have that ∂R is a polygonal loop consisting of four line segments from the point (a, c) to (b, c), then from (b, c) to (b, d), then (b, d) to (a, d), and finially from (a, d) to (a, c).

Theorem 4.8. *Suppose $f(z)$ is holomorphic in a region U containing R. Then, we have*

$$\int_{\partial R} f dz = 0.$$

Proof. We bisect the rectangle R by choosing points $\frac{a+b}{2}$ and $\frac{c+d}{2}$. We obtain four rectangles R_i, $i = 1, \ldots, 4$ as in Figure 4. Since the sides of the rectangles R_i that overlap appear once with one

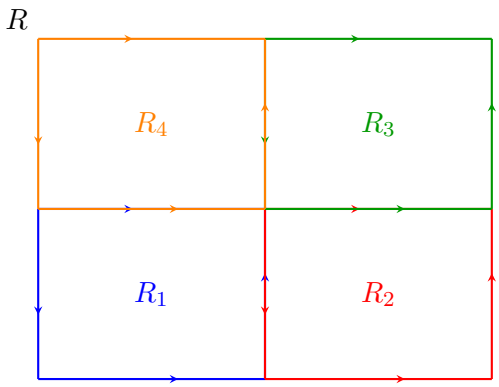


FIGURE 5. Bisection of rectangle R into sub-rectangles R_1, R_2, R_3, R_4.

orientation, and once with the opposite orientation, we have the quality

$$\int_{\partial R} f dz = \int_{\partial R_1} f dz + \int_{\partial R_2} f dz + \int_{\partial R_3} f dz + \int_{\partial R_4} f dz. \tag{9}$$

This also implies that at least one of the the values $\int_{\partial R_i} f dz$ should be larger than the average $\frac{1}{4}\int_{\partial R} f dz$ in modulues, i.e. we have

$$|\int_{\partial R_i} f dz| \quad \geq \quad \frac{1}{4}|\int_{\partial R} f dz|.$$

We can rename this rectangle R_i as R^1. We inductively iterate the process, each time bisecting R^n, and obtaining a corresponding R^{n+1} such that

$$|\int_{\partial R^{n+1}} f dz| \quad \geq \quad \frac{1}{4}|\int_{\partial R^n} f dz|$$

holds. Therefore, we obtain a sequence of rectangles R^n with the property that

$$|\int_{\partial R^n} f dz| \quad \geq \quad \frac{1}{4^n}|\int_{\partial R} f dz|, \tag{10}$$

and such that $R := R^0 \supset R^1 \supset \cdots R^n \supset \cdots$. The sets R^n satisfy the hypotheses of Corollary 3.34, and therefore there exists a point $z^* \in \bigcap_{n=0}^{\infty} R^n$. Since $z^* \in R \subset U$, and U is open, we can choose a $\delta' > 0$ such that the disc $D(z^*, \delta')$ is contained in U, ad therefore f is holomorphic on this disc. By definition of holomorphic function, for a given $\epsilon > 0$ we can find $\delta > 0$, with $\delta < \delta'$, such that for all $z \in D(z^*, \delta)$ we have

$$|\frac{f(z) - f(z^*)}{z - z^*} - f'(x^*)| \quad < \quad \epsilon,$$

which implies

$$|f(z) - f(z^*) - f'(z^*)(z - z^*)| \quad < \quad \epsilon|z - z^*|.$$

For n large enough, we will have $R^n \subset D(x^*, \delta)$, since the sides of R^n converge to zero as n grows to ∞. By the fact that both 1 and z are the derivatives of some holomorphic function, we have

that

$$\int_{\partial R^n} dz = 0, \tag{11}$$

$$\int_{\partial R^n} z dz = 0. \tag{12}$$

Therefore, we find

$$\begin{aligned} |\int_{\partial R^n} f(z)dz| &= |\int_{\partial R^n} [f(z) - f(z^*) - (z-z^*)f'(z^*)]dz| \\ &\leq \int_{\partial R^n} |f(z) - f(z^*) - (z-z^*)f'(z^*)||dz| \\ &\leq \int_{\partial R^n} \epsilon |z-z^*||dz|. \end{aligned}$$

Since $z \in \partial R^n$ implies that $|z - z^*| \leq d_n$, the diagonal of R^n, we obtain that the integral satisfies

$$|\int_{\partial R^n} f(z)dz| \leq d_n L_n \epsilon,$$

where L_n is the perimeter of R^n. Since we had set $R^0 = R$, we find that $d_n \leq 2^{-n} d_0$ and $L_n \leq 2^{-n} L_0$. Therefore one gets

$$|\int_{\partial R^n} f(z)dz| \leq 4^{-n} d_0 L_0 \epsilon.$$

From (10) we therefore deduce that

$$|\int_{\partial R} f(z)dz| \leq d_0 L_0 \epsilon.$$

Since ϵ was arbitrary, it follows that $\int_{\partial R} f(z)dz = 0$. □

There is a stronger version of the previous result, in the sense that it has weaker assumptions.

Theorem 4.9. *Let f be holomorphic on the neighborhood U of a rectangle R, except at finitely many points ζ_i, $i = 1, \ldots, n$, in the interior of R. Suppose that*

$$\lim_{z \to \zeta_i} (z - \zeta_i) f(z) = 0,$$

for each $i = 1, \ldots, n$. Then we have

$$\int_{\partial R} f dz = 0.$$

Proof. If $n > 1$, we can subdivide the rectangle R in subrectangles in such a way that each of them contains one of the points where f is not holomorhic. Then, as in the proof of Theorem 4.8 the sum of the integrals would give the integral over ∂R. Therefore, we can consider the case where $n = 1$ without loss of generality. Let $\epsilon > 0$ be arbitrarily chosen. Upon subdividing R multiple times, we can obtain a configuration where the singularity ζ belongs to a subrectangle R' contained in the disc $D(\zeta, \delta)$, where δ is such that $|(z-\zeta)f(z)| < \epsilon$ for each $z \in D(\zeta, \delta)$, while all other subrectangles do not contain any singularity. One has, applying Theorem 4.8 to all subrectangles except R'

$$\begin{aligned} \int_{\partial R} f(z)dz &= \sum_{R_i} \int_{\partial R_i} f(z)dz + \int_{\partial R'} f(z)dz \\ &= \int_{\partial R'} f(z)dz. \end{aligned}$$

However, by the choice of δ and $R' \subset D(\zeta, \delta)$, for every $z \in R'$ it holds

$$|f(z)| \leq \frac{\epsilon}{|z-\zeta|}.$$

Then, we find the estimate

$$\int_{\partial R} f(z)dz \leq \epsilon \int_{\partial R'} \frac{|dz|}{|z-\zeta|}.$$

Since one can see that $\int_{\partial R'} \frac{|dz|}{|z-\zeta|} < 8$, we find that $\int_{\partial R} f(z)dz < 8\epsilon$ for each $\epsilon > 0$, and therefore the integral must vanish. □

We now consider the case of a disc $\Delta := D(z_0, \rho)$.

Theorem 4.10. *Suppose f is holomorphic on Δ. Then, for any closed curve (i.e. loop) $\gamma \subset \Delta$ we have*

$$\int_\gamma f(z)dz = 0.$$

Proof. We define the function

$$F(z) = \int_\sigma f(z)dz,$$

where σ is a polygonal curve consisting of segments that are horizontal or vertical. Applying Theorem 4.8, we find that F is well defined, since the integral does not depend on what sides of a rectangle we use for the path σ.

Let σ be the path from the point z_0 to z obtained by joining the two paths: $z_1(t) = ((1-t)x_0 + tx, y_0)$ and $z_2(t) = (x, (1-t)y_0 + ty)$, both of which are contained in the disc Δ, since this is convex. As in the proof of Theorem 4.5 we find that $\frac{\partial F}{\partial y} = if(z)$. Similarly, choosing σ to be the union of thwo arcs, a vertical one and then a horizontal one, we find that $\frac{\partial F}{\partial x} = f(z)$. This implies that $f(z)dz$ is an exact differential. By Theorem 4.5, the integral along the loop γ can be computed by considering any loop. In particular, choosing γ to be the constant loop, we see that the integral vanishes. □

Theorem 4.11. *Suppose f is holomorphic on $\Delta' := \Delta - \{\zeta_1, \ldots, \zeta_n\}$, where Δ is as above. If f satisfies the condition*

$$\lim_{z \to \zeta_j} (z - \zeta_j) f(z) = 0,$$

for all $j = 1, \ldots, n$, then

$$\int_\gamma f(z)dz = 0$$

holds for any loop $\gamma \subset \Delta'$.

Proof. By considering a polygonal curve σ consisting of at most four arcs, we can join z_0 with a generic point z without passing through ζ_j and with the property that each arc is horizontal/vertical (with the last arc being either horizontal or vertical depending on the partial derivative that is being computed). Applying Theorem 4.9 one can see that the primitive function $F(z)$ as defined in the proof of Theorem 4.10 is well defined in the sense that the choice of path does not affect the computation of $F(z)$, and moreover $\frac{\partial F}{\partial x} = f$ and $\frac{\partial F}{\partial y} = if$. This completes the proof by applying Theorem 4.5. □

We now introduce the notion of *index*, or *winding number*, of a closed curve γ with respect to a point $\zeta \in \mathbb{C}$. This concept formalizes what we intuitively think of being the number of times that a curve winds around a given point ζ. We have the following preliminary result.

Lemma 4.12. *Let γ be a loop such that $\zeta \notin \gamma$. Then, we have*

$$\int_\gamma \frac{dz}{z-\zeta} = 2k\pi i, \tag{13}$$

for some $k \in \mathbb{Z}$.

Proof. Let $z(t)$ be a parametrization of γ, where $t \in [\alpha, \beta]$. Define the auxiliary function

$$h(t) = \int_\alpha^t \frac{z'(s)}{z(s)-\zeta} ds.$$

By construction, we have that $h'(t) = \frac{z'(t)}{z(t)-\zeta}$ whenever $z'(t)$ is continuous, which happens everywhere except at possibly finitely many points. Therefore, the function $g(t) = e^{-h(t)}(z(t)-\zeta)$ has derivative given by

$$\begin{aligned} g'(t) &= -h'(t)e^{-h(t)}(z(t)-\zeta) + e^{-h(t)}z'(t) \\ &= 0, \end{aligned}$$

except for possibly finitely many parameter points. Since $g(t)$ is continuous, it follows that it is a constant. If we denote this constant by k, and observe that $h(\alpha) = 0$ by definition, we see that

$$e^{h(t)} = \frac{1}{k}(z(t)-\zeta),$$

for every t, and $k = z(\alpha) - \zeta$. Therefore,

$$e^{h(t)} = \frac{z(t)-\zeta}{z(\alpha)-\zeta}.$$

However, since γ is a loop, it follows that $z(\beta) = z(\alpha)$, and therefore $e^{h(\beta)} = 1$, which implies that $h(\beta)$ is an integer multiple of $2\pi i$. □

The lemma allows us to pose the following definition.

Definition 4.13. Let γ be a closed loop, and let $\zeta \notin \gamma$. Then, we define the *index*, or *winding number*, of γ with respect to ζ as the integer

$$n(\gamma, \zeta) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\zeta}.$$

Proposition 4.14. *The winding number has the following properties.*

(i) $n(-\gamma, \zeta) = -n(\gamma, \zeta)$.
(ii) *If γ is contained in a disc D such that $\zeta \notin D$, then $n(\gamma, \zeta) = 0$.*
(iii) *The curve γ determines complementary regions of $\mathbb{C}$, only one of which is unbounded. As a function of ζ, the winding number $n(\gamma, \zeta)$ is constant in each of the regions determined by γ, and it is zero on the unbounded region.*

Proof. Both (i) and (ii) follow directly from definitions and our previous results. We prove (iii). Since γ is closed, its complement is open. By Theorem 3.62, we can decompose $\mathbb{C} - \gamma$ into components. Since $\mathbb{C} - \gamma$ is open, each of the components are open by Theorem 3.63, and therefore are regions of $\mathbb{C}$. Since γ is the continuous image of a compact, it is compact. Therefore, by

Theorem 3.45 γ is bounded. It follows that if we consider $\mathbb{C}_\infty$ instead of $\mathbb{C}$, ∞ belongs to $\mathbb{C}_\infty - \gamma$, and therefore it is an accumulation point of only one of the components of $\mathbb{C} - \gamma$. This implies that there is only one unbounded component. To see that $n(\gamma, \zeta)$ is constant on the regions of $\mathbb{C} - \gamma$ we can proceed as follows. Any two points ζ_1 and ζ_2 of a region can be joined by a polygonal curve in the same region—hence not intersecting γ—by Theorem 3.60. Therefore, it is sufficient to prove that $n(\gamma, \zeta_1) = n(\gamma, \zeta_2)$ if if ζ_1 and ζ_2 can be joined by a line segment that does not intersect γ. Then, we have that the quantity

$$\frac{z-\zeta_1}{z-\zeta_2} \quad = \quad e^{i(arg(z-\zeta_1)-arg(z-\zeta_2))}\left|\frac{z-\zeta_1}{z-\zeta_2}\right|,$$

is never real and non-positive for $z \notin \gamma$. Therefore the principal branch of the logarithm $\log \frac{z-\zeta_1}{z-\zeta_2}$ is holomoprhic on the segment joining ζ_1 and ζ_2. Since the derivative of $\log \frac{z-\zeta_1}{z-\zeta_2}$ is $\frac{1}{z-\zeta_1} - \frac{1}{z-\zeta_2}$, it follows that

$$\int_\gamma (\frac{1}{z-\zeta_1} - \frac{1}{z-\zeta_2})dz \quad = \quad 0,$$

from which $n(\gamma, \zeta_1) = n(\gamma, \zeta_2)$ follows. To prove that $n(\gamma, \zeta)$ is zero on the unbounded compoent of $\mathbb{C} - \gamma$, we consider that the boundedness of γ implies that we can find a disc D which contains γ, and such that $\mathbb{C} - D \subset U$, where U is the unbounded region of $\mathbb{C} - \gamma$. Applying (ii), we have that on any point $\zeta \in (C - D) \cap U \neq \emptyset$, $n(\gamma, \zeta) = 0$. Applying the first part of the proof of (iii) one can therefore deduce that $n(\gamma, \zeta) = 0$ for every $\zeta \in U$. □

While the intuitive notion of being "on the outside" of a closed curve is not well posed, the previous result gives a way of formally saying this. A point that belongs to the unbounded region determined by a curve is "on the outside" of the closed curve. And further, the fact that the winding number is zero formalizes the intuitive way of thinking that the curve does not "wrap around" a point on the outside.

Lemma 4.15. *Let z_1 and z_2 be two points on a loop γ which does not pass through the origin. Let γ_1 denote the section of γ that goes from z_1 to z_2, and let γ_2 be the section that goes from z_2 to z_1. Suppose that z_1 is in the lower-half plan e $\mathbb{H}^-$, and z_2 is in the upper-helf plane $\mathbb{H}^+$. If $\gamma_1 \cap \mathbb{R}^- = \gamma_2 \cap \mathbb{R}^+ = \emptyset$, then we have $n(\gamma, 0) = 1$.*

Proof. Let L_1 and L_2 denote the lines through 0 and z_i, $i = 1, 2$, respectively. Let $C = \partial D$ be a circle about the origin, where D is the disc whose boundary is C. Let ζ_i denote the point on $C \cap L_i$. If we travel on C from ζ_1 to ζ_2 in the anticlockwise manner, we do not intersect the negative real axis, and when going from ζ_2 to ζ_2 we do not intersect the positive real axis. We let these paths be C_1 and C_2 respectively. Denote the directed line segment from z_i to ζ_i by δ_i, and define the paths $\sigma_1 = \gamma_1 + \delta_2 - C_1 - \delta_1$ and $\sigma_2 = \gamma_2 + \delta_1 - C_2 - \delta_2$. After appropriate cancellation, and using (i) in Proposition 4.14, we find that $n(\gamma, 0) = n(C, 0) + n(\sigma_1, 0) + n(\sigma_2, 0)$. By construction, the origin is in the unbounded region for both σ_1 and σ_2, and therefore by (iii) of Proposition 4.14 we have $n(\sigma_i, 0) = 0$ and $n(\gamma, 0) = n(C, 0) = 1$, where the last equality comes from Example 4.7. □

Theorem 4.16. *Let f be a function which is holomorphic on an open disc D, and let γ denote a loop in D and $a \notin \gamma$. Then we have*

$$n(\gamma, a)f(a) \quad = \quad \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz. \tag{14}$$

Proof. Define the function

$$F(z) \quad = \quad \frac{f(z)-f(a)}{z-a},$$

with the understanding that $f(a)$ is set to zero if $a \notin D$. By construction, F is holomorphic everywhere on D, except posssibly on the point $z=a$, where it satisfies the condition

$$\lim_{z\to a}(z-a)F(z) \quad = \quad 0.$$

Therefore, applying Theorem 4.11, we find that

$$\int_\gamma F(z)dz \quad = \quad 0,$$

from which we find

$$\begin{aligned}\int_\gamma \frac{f(z)}{z-a}dz &= f(a)\int_\gamma \frac{1}{z-a}dz \\ &= 2\pi i f(a) n(\gamma, a).\end{aligned}$$

The result follows. □

If take γ to be such that $n(\gamma, a) = 1$, then we have a representation for the value $f(a)$ as

$$f(a) \quad = \quad \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz. \tag{15}$$

Changing variables, we can write the Equation (15) as a function of z as

$$f(z) \quad = \quad \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta. \tag{16}$$

Equation (16) is generally called *Cauchy's integral formula.* We note that the result would be applicable even if f is not holomorphic on finitely many points, as long as the conditions of Theorem 4.11 are satisfied, and γ does not contain any of the points where f is not holomorphic.

We are finally in the position to show that holomorphic functions have derivatives of all orders.

Theorem 4.17. *Let f be holomorphic on a region U. Then, f can be differentiated infinitely many times, and its n^{th} derivative is given by*

$$f^{(n)}(z) \quad = \quad \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta, \tag{17}$$

where C is the boundary of a disc centered at z and contained in U.

Proof. Observe that since U is a region is in particular open, and therefore for any $z \in U$ such disc centered at z contained in U would always exist. We prove the following fact, from which the result would follow: For any continuous $\phi(z)$ on a closed loop $\gamma \subset U$, the function

$$F_n(z) \quad := \quad \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n}d\zeta \tag{18}$$

is differentiable with derivative given by $F_n'(z) = nF_{n+1}(z)$.

First, we show that $F_1(z)$ is continuous. For $z_0 \notin \gamma$, $z_0 \in U$, we choose $\delta > 0$ such that $D(z_0, \delta) \subset U$, and $D(z_0, \delta) \cap \gamma = \emptyset$. In particular, we find that $|\zeta - z| > \frac{\delta}{2}$ for all $\zeta \in \gamma$, and all $z \in D(z_0, \frac{\delta}{2})$. Therefore, we have

$$\begin{aligned} |F_1(z) - F_1(z_0)| &= \left|(z - z_0)\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)(\zeta - z_0)} dz\right| \\ &< |z - z_0|\frac{2}{\delta^2}\int_\gamma |\phi(\zeta)||d\zeta|. \end{aligned}$$

Continuity of F_1 then follows. Moreover, from the construction above, we also see that

$$\frac{F_1(z) - F_1(z_0)}{z - z_0} = \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)(\zeta - z_0)} d\zeta.$$

Taking the limit $z \to z_0$, we see that the right-hand-side above tends to $F_2(z_0)$ by applying the continuity of F_1 to the function $\frac{\phi(\zeta)}{\zeta - z_0}$, therefore showing that the derivative of F_1 on each point z_0 of U is $F_2(z_0)$. To complete the proof, we now proceed by induction. Suppose that the result is true for some integer $n - 1 \geq 1$. In other words, for any continuous function, we have that the corresponding F_{n-1} satisfies $F'_{n-1}(z) = (n-1)F_n(z)$. Then, applying the identity

$$\frac{1}{(\zeta - z)^n} = \frac{1}{(\zeta - z)^{n-1}(\zeta - z_0)} + \frac{z - z_0}{(\zeta - z)^n(\zeta - z_0)},$$

we can rewrite the difference $F_n(z) - F_n(z_0)$ as

$$F_n(z) - F_n(z_0) = [\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)} d\zeta - \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^n} d\zeta] + (z - z_0)\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n(\zeta - z_0)} d\zeta.$$

Let us consider the term in square brackets above. This can be rewritten as

$$\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)} d\zeta - \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^n} d\zeta = \int_\gamma \frac{\frac{\phi(\zeta)}{\zeta - z_0}}{(\zeta - z)^{n-1}} d\zeta - \int_\gamma \frac{\frac{\phi(\zeta)}{\zeta - z_0}}{(\zeta - z_0)^{n-1}} d\zeta,$$

where $\frac{\phi(\zeta)}{\zeta - z_0}$ is a continuous function of ζ on γ, and therefore we can apply the inductive hypothesis to it to find that the function $\frac{\frac{\phi(\zeta)}{\zeta - z_0}}{(\zeta - z)^{n-1}}$ is differentiable, hence it is continuous. Similarly to how we proceeded for the case $n = 1$, we have that $(z - z_0)\int_\gamma \frac{1}{(\zeta - z)^n(\zeta - z_0)}$ tends to zero as z goes to z_0. Moreover, by the continuity of $\frac{\frac{\phi(\zeta)}{\zeta - z_0}}{(\zeta - z)^{n-1}}$, we also have

$$\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)} d\zeta - \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^n} d\zeta \to 0,$$

as $z \to z_0$. It follows that $F_n(z) \to F_n(z_0)$ as $z \to z_0$ since

$$\begin{aligned} |F_n(z) - F_n(z_0)| &\leq \left|\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)} d\zeta - \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^n} d\zeta\right| + \left|(z - z_0)\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n(\zeta - z_0)} d\zeta\right| \\ &\to 0. \end{aligned}$$

So F_n is continuous for every continuous function used in the integrand. We now consider the quotient $\frac{F_n(z)-F_n(z_0)}{z-z_0}$. We have

$$\begin{aligned}
&\frac{F_n(z)-F_n(z_0)}{z-z_0}\\
&= \frac{1}{z-z_0}\Big[\int_\gamma \frac{\phi(\zeta)}{(\zeta-z)^{n-1}(\zeta-z_0)}d\zeta - \int_\gamma \frac{\phi(\zeta)}{(\zeta-z_0)^n}d\zeta\Big] + \int_\gamma \frac{\phi(\zeta)}{(\zeta-z)^n(\zeta-z_0)}d\zeta\\
&= \frac{1}{z-z_0}\Big[\int_\gamma \frac{\frac{\phi(\zeta)}{\zeta-z_0}}{(\zeta-z)^{n-1}}d\zeta - \int_\gamma \frac{\frac{\phi(\zeta)}{\zeta-z_0}}{(\zeta-z_0)^{n-1}}d\zeta\Big] + \int_\gamma \frac{\frac{\phi(\zeta)}{\zeta-z_0}}{(\zeta-z)^n}d\zeta.
\end{aligned}$$

Then, taking the limit as $z \to z_0$ we get

$$\lim_{z\to z_0}\frac{F_n(z)-F_n(z_0)}{z-z_0} = (n-1)\int_\gamma \frac{\phi(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta + \int_\gamma \frac{\phi(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta,$$

where the first term is due to the inductive hypothesis applied to $\frac{\phi(\zeta)}{\zeta-z_0}$, and the second term is due to the fact that we proved that F_n is continuous for each continuous integrand, such as $\frac{\phi(\zeta)}{\zeta-z_0}$. The proof is complete. □

We now prove some important consequences of the previous result.

Theorem 4.18 (Morera's theorem)**.** *If f is continuous on a region U, and if $\int_\gamma f(z)dz = 0$ for all closed curves γ in U, then f is holomorphic in U.*

Proof. Since $\int_\gamma f(z)dz = 0$ for every loop γ, from Theorem4.5 we find a holomorphic function F such that $f = F'$. By Theorem 4.17 it follows that f itself is holomorphic. □

Theorem 4.19 (Liouville's theorem)**.** *A function that is holomorphic and such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$ is constant.*

Proof. Let $z_0 \in \mathbb{C}$, and let $D := D(z_0, r)$ be a disc of radius r. Applying (17) to the boundary $C = \partial D$, we find the estimate

$$\left|f^{(n)}(z_0)\right| \leq \frac{Mn!}{r^n}.$$

In particular, for the first derivative f' of f, we find that $|f'(z)| \leq \frac{M}{r}$ for all $r > 0$. It follows that $f'(z) = 0$ for all $z \in \mathbb{C}$, from which it follows that the function f is constant. □

Theorem 4.20 (Fundamental Theorem of Algebra)**.** *Every polynomial p of degree larger than zero has a root in $\mathbb{C}$.*

Proof. If p has no roots, then the function $f(z) = \frac{1}{p(z)}$ is holomorphic everywhere. Moreover, $\lim_{|z|\to\infty} f(z) = 0$ since p has degree larger than zero. Let D be a disc of radius r, such that $|f(z)| \leq M_1$ for $|z| > r$. Applying Theorem 3.45 and Theorem 3.27 we find that the function $|f|$ attains its maximum on $\bar{D}$, which we denote M_2. It follows that $|f(z)| \leq \max\{M_1, M_2\}$ for all $z \in \mathbb{C}$. Therefore, applying Theorem 4.17 it follows that f is constant. However, this contradicts the fact that p has degree larger than zero, and p must have some root in $\mathbb{C}$. □

We now consider the notion of removable singularities. The following result shows when we can remove a singularity by associating a value to it in a holomorphic fashion.

Theorem 4.21. *Suppose that f is holomorphic in the region $\Omega' = \Omega - \{a\}$, where Ω is a region. Then, f can be extended to a holomorphic function on Ω if and only if $\lim_{z\to a}(z-a)f(z) = 0$. If such an extension exists, it is unique.*

Proof. If f can be extended to a holomorphic function $\hat{f}$ on Ω, from the fact that $\hat{f}$ is continuous at a, it follows that $\lim_{z\to a}(z-a)\hat{f}(z) = 0$, which implies the required equality since $\hat{f} = f$ on Ω'. Moreover, by continuity (and uniqueness of limits), we also have that the extension is uniquely determined.

Conversely, we want to show that if $\lim_{z\to a}(z-a)f(z) = 0$, then we can extend f to Ω. Let $D(a,\rho)$ be a disc around a contained in Ω, which can be found since Ω is a region. Let $C = \partial D(a,\rho)$ be the circle boundary of the disc. The function defined by the integral $z \mapsto \int_C \frac{f(\zeta)}{\zeta - z}d\zeta$ is holomorphic on the whole $D(a,\rho)$, as shown in the proof of Theorem 4.17, therefore we can set $\hat{f}(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta$ for every $z \in D(a,\rho)$. By Cauchy's formula, which is applicable due to the equality $\lim_{z\to a}(z-a)f(z) = 0$ as noted in the discussion before Theorem 4.17, for every $z \in D(a,\rho) - \{a\}$ we have

$$f(z) \quad = \quad \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta.$$

Therefore, $\hat{f}(z) = f(z)$ for every $z \in D(a,\rho) - \{a\}$. This means $\hat{f}$ extends f holomorphically. □

The procedure of using the integral representation of the function allows us to define a Taylor expansion of a holomorphic function.

Theorem 4.22. *Let f be holomorphic on the region Ω, and let $a \in \Omega$. Then, for each $n \in \mathbb{N}$ we can find a holomorphic function f_n such that*

$$f(z) \quad = \quad \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(z-a)^k + f_n(z)(z-a)^n. \tag{19}$$

Proof. By induction, we show that for each n we can find holomorphic functions f_i such that

$$f(z) \quad = \quad f(a) + f_1(a)(z-a) + \cdots + f_n(z)(z-a)^{n+1},$$

and $f_n(a) = \frac{f^{(n)}(a)}{n!}$, from which the result would follow. Define the function

$$F(z) \quad = \quad \frac{f(z) - f(a)}{z - a}.$$

This is defined on $\Omega - \{a\}$, on which it is holomorphic. Since $\lim_{z\to a}(z-a)F(z) = 0$, we can apply Theorem 4.21 to extend F to the whole Ω holomorphically. Moreover, since $\lim_{z\to a} F(z) = f'(a)$, it follows that the value of the extended F at a is given by $f'(a)$. Let $f_1(z)$ denote the extended function just obtained. Then, we have just shown that $f(z) = f(a) + f_1(z)(z-a)$, which gives us the base of induction for our proof. Suppose the result has been proved for some $n \geq 1$. Define

$$F(z) \quad = \quad \frac{f_n(z) - f_n(a)}{z - a}.$$

As before, extend F to the whole Ω, and by construction we will have that $F(a) = f^{(n+1)}(a)$. We denote the extension F by f_n. Then, it holds that $f_n(z) = f_n(a) + f_{n+1}(z)(z-a)$. Moreover, for each $0 \leq k \leq n$ we have

$$f_k(z) \quad = \quad f_k(a) + f_{k+1}(z-a),$$

where $f_0 := f$. By substituting we derive the equality

$$f(z) \quad = \quad f(a) + f_1(a)(z-a) + \cdots + f_{n+1}(z)(z-a)^{n+1}.$$

Differentiating $n+1$ times and evaluating at a gives $f^{(n+1)}(a) = (n+1)!f_{n+1}(a)$. □

Let n be the smalles $n \in \mathbb{N}$ such that $f^{(k)}(a) = 0$ for all $0 \leq k < n$. Then, we can write by Theorem 4.22

$$f(z) \quad = \quad f_n(z)(z-a)^n,$$

for some holomorphic f_n, such that $f_n(a) \neq 0$. By convention, if $f^n(a) = 0$ for each $n \in \mathbb{N}$, we set this minimum number to be ∞. In this case we would have that $f(z) = f_n(z)(z-a)^n$ holds for every n, and $f_n(a) = 0$. We want to show that in this latter case f vanishes identically over the whole region Ω.

Theorem 4.23. *Suppose f as above is such that $f^{(n)}(a) = 0$ for all $n \in \mathbb{N}$. Then f vanishes in the whole region Ω.*

Proof. In the Cauchy formula of the holomorphic function f_n in a disc $D(a,\rho) \subset \Omega$, with $C = \partial D(a,\rho)$, we have

$$f_n(z) \quad = \quad \frac{1}{2\pi i}\int_C \frac{f_n(\zeta)}{\zeta - a}d\zeta.$$

Applying (19), we have

$$f_n(z) \quad = \quad \frac{f(z)}{(z-a)^n} - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(z-a)^{k-n}.$$

It therefore follows that

$$f_n(z) \quad = \quad \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-a)^n(\zeta-z)}d\zeta - \sum_{k=0}^{n-1}\frac{1}{2\pi i}\int_C \frac{f^{(k)}(a)}{k!}(\zeta-a)^{k-n}(\zeta-z)^{-1}.$$

We claim that $\int_C \frac{f^{(k)}(a)}{k!}(\zeta-a)^{k-n}(\zeta-z)^{-1} = 0$ for each $k = 0, \ldots, n-1$. We set

$$F_\nu(a) \quad = \quad \int_C \frac{d\zeta}{(\zeta-a)^\nu(\zeta-z)}.$$

We have $F_1(a) = \int_C [\frac{1}{\zeta-a} - \frac{1}{\zeta-z}]d\zeta = 0$. Applying the same procedure as in the proof of Theorem4.17, we see that $F_\nu(a) = F_1(\nu)(a)/\nu!$, so that $F_\nu(a) = 0$ for each ν. Then, we have

$$f_n(z) \quad = \quad \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-a)^n(\zeta-z)}d\zeta. \tag{20}$$

Since C is compact, and f is continuous, by Theorem 3.27 $f(C)$ is compact, and therefore there it attains its maximum value. Let $M = \max_{z\in C} f(z)$. From (20) we therefore find the estimate

$$|f_n(z)| \quad \leq \quad \frac{M}{\rho^{n-1}(\rho - |z-a|)},$$

for each z such that $|z-a| < \rho$. SInce $f(z) = f_n(z)(z-a)^n$, we have the estimate

$$|f(z)| \quad \leq \quad \frac{|z-a|^n}{\rho^n}\frac{M\rho}{\rho - |z-a|} \xrightarrow{n\to\infty} 0.$$

Therefore, f is identically zero inside the disc $D(a,\rho) \subset \Omega$. Also, all the derivatives of f vanish inside $D(a,\rho)$. To complete the proof, we conside the disjoint decomposition of $\Omega = E_1 \sqcup E_2$ where

E_1 is the set of points of Ω such that f and all its derivatives vanish, while E_2 is the complement of E_1. The set E_1 is open since we have shown above that if $a \in E_1$ a whole disc lies inside E_1. Moreover, E_2 is open since f and all its derivatives are continuous, so if $f^{(n)}(z_0) \neq 0$ for some $n = 0, 1, \ldots,$, we would have that $f^{(n)}(z) \neq 0$ for all z sufficiently close to z_0. Then, since Ω is connected, we either have $E_1 = \emptyset$ or $E_2 = \emptyset$. However, E_1 contains a, and therefore we must have $E_2 = \emptyset$, which completes the proof. □

It follows that if a holomorphic f is not identically zero on the region Ω, there must be a minimum $n \in \mathbb{N}$ such that $f(z) = f_n(z)(z-a)^n$ holds for some holomorphic n, where we just set by convention $f_0 = f$ if a is not a zero of f. Such natural number n is called the *order of the zero*. This means that holomorphic functions behave like polynomials around zeros. In particular, unless the function vanishes identically on a region, the zeros of a holomorphic function are isolated. We therefore obtain the following.

Corollary 4.24. *Let f and g be holomorphic functions on a region Ω. Suppose that $f(z) = g(z)$ for a set of points in Ω that has an accumulation point. Then $f = g$ identically on Ω.*

Suppose now that f is holomorphic on a neighborhood of a point a, except at a. For instance, on the pierced disc $D(a, \rho) - \{a\}$. Such point is called an isolated singularity. We have considered *removable* singularities in Theorem 4.21. Suppose now $\lim_{z \to a} f(z) = \infty$. By using the Rieman sphere, we can set $f(a) = \infty$. Suppose $\delta \leq \rho$ is such that in $D(a, \delta)$ we have $f(z) \neq 0$. Then, we can define $g(z) = \frac{1}{f(z)}$, and in $D(a, \delta)$ is holomoprhic and it has a zero in a (or better to say its extension to $z = a$ is holomorphic and with a zero). Since g does not vanish identically on $D(a, \delta)$, it follows that the zero of g has an order n. We call this number the *order of the pole* of f at a. A function that is holomorphic in a region Ω except for its poles, is said to be *meromorphic*.

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