LECTURE NOTES MATH 2240 - LINEAR ALGEBRA

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1. Introduction

Linear algebra is the branch of mathematics that studies systems of linear equations, and more generally vector spaces, matrices and linear transformations between them. It is important in formulating several real world problems in a first order approximation. By linear, it is meant that there are no terms that are quadratic, cubic etc.

Example 1.1. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = 3x$ is linear, while the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = 3x^2$ is not linear (it is quadratic!).

Important applications of linear algebra can be found in artificial intelligence, where several algorithms use linear maps as central components. Morevoer, in machine learning, neural networks are obtained by concatenating several linear maps (deep learning) with certain objects called activation functions in between. More advanced topics in mathematics such as functional analysis deal with linear algebra applied to spaces of functions. Linear algebra is also a very common tool to model real world systems.

Example 1.2. Consider a simplified economic system with 3 sectors: Coal (C), Electric (E) and Steel (S). Each sector produces an output which can be possibly be bought by one other sector. For instance, the output of the Electric sector can be distributed among Coal and Steel in some fractions. Similarly for the other sectors. Suppose the outputs are distributed according to Table 1 If by p_C, p_E and p_S we denote the price of the outputs of the three sectors, then in order to find the equilibrium prices (i.e. when a sector sells as much as it gains), we find the system of equations

$$\begin{cases} p_C = 0.1p_C + 0.5p_E + 0.4p_S \\ p_E = 0.3p_C + 0.4p_E + 0.3p_S \\ p_S = 0.6p_C + 0.2p_E + 0.2p_S \end{cases}$$

which we would need to solve. Using linear algebra we will be able to solve such a system of equations.

These notes are based on the textbook [2], which has been used to teach this course. More advanced references on the subject are [1, 3].

Coal	Electric	Steel	Purchased by
0.1	0.3	0.6	Coal
0.5	0.4	0.2	Electric
0.4	0.3	0.2	Steel

Table 1. Simple linear economy with three sectors

2. Systems of linear equations

2.1. **Generalities.** An equation is said to be linear if the only operations applied to the variables appearing in it are multiplicaations by numbers and additions. In other words, a linear equation in the variables $x_1, \dots x_n$ is an equation that can be written as

$$a_1 x_1 + \cdots + a_n x_n = b, \tag{1}$$

for some real or complex numbers $a_1, \dots a_n, b$, which are called coefficients and are known a priori (they are determined by the problem at hand).

Example 2.1. *An example is the equation*

$$3x_1 + 2x_2 = 7.$$

A non-example (why?) is the equation

$$2\cos(x_1) + x_2^2 = 0.$$

A system of linear equations is a collection of several linear equations as above. For instance, the linear economic example above (Example 1.2). Another example is

$$\begin{cases} 2x_1 + \sqrt{3}x_2 = 2 \\ x_1 + 7x_2 = 0. \end{cases}$$

In such a case we want to find values of x_1 and x_2 such that both equations are satisfied *simultaneously*. Here, the fact that we want to solve both equations at the same time is fundamental. There might be cases where no solution exists, and we would also like to be able to know when this happens.

In general, for systems of linear equations, there might be two scenarios. The first one, is when no solutions exist. The second one is when there are solutions. In the second case, we also want to understand when the solution is unique, or when there are many solutions and how we can describe them in order to obtain them all. We call *solution set* the set of all possible solutions of the system. If there are no solutions, then we say that the solution set is empty.

The geometric meaning of a linear equation as $2x_1 + \sqrt{3}x_2 = 2$ is that of a line lying in the plane. Therefore, a solution to a system of two linear equations in two variables is the intersection of the two lines that correspond to the given equations. This helps us understand when the system has a single solution, or when it has infinitely many solutions, and when it has no solution at all.

Question 2.2. How do you characterize the three cases mentioned above?

More generally, a system of equation has a unique solution, or infinitely many solutions, or no solution at all. If a system has solutions, then it is said to be *consistent*, while if it has no solutions it is said to be *inconsistent*.

Systems of linear equations can be more compactly represented by their matrix notation. A matrix is a table of numerical values. The table has n rows and m columns and we index the numerical values depending on their position in the table. For example, the entry a_{13} indicates the number lying in the first row, and third column. To the system written above, we associate the matrix

$$A = \begin{bmatrix} 2 & \sqrt{3} \\ 1 & 7 \end{bmatrix}.$$

In this notation, we can write the system as

$$Ax = B,$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

We are therefore induced to introduce the following product between a matrix (a table) and a vector (a column): $(Ax)_{i1} = A_{i1}x_1 + Ai2x_2$, where (Ax) has entries of type $i1$ with 1 fixed because it is a column.

The matrix A can be *augmented* to contain also the coefficients of B:

$$\tilde{A} = \begin{bmatrix} 2 & \sqrt{3} & 2 \\ 1 & 7 & 0 \end{bmatrix}.$$

2.2. **Solving linear systems.** The idea of solving a linear system is to use the first equation to express the variable x_1 in terms of the other variables, e.g. x_2 in the case of two equations and with two variables. Then, proceeding like this for all equations, we can finally have an equation in a single variable, which is easy to solve. Plugging back the value in the previous expressions we obtain a solution. This intuitive idea is illustrated in the following.

Example 2.3. We want to use this intuition to solve the system

$$\begin{cases} 2x_1 + \sqrt{3}x_2 = 2 \\ x_1 + 7x_2 = 0. \end{cases}$$

We strt by rewriting x_1 in terms of x_2 using the second equation (we could start from the first, the second is simpler!). We have that $x_1 + 7x_2 = 0$, meaning that $x_1 = -7x_2$. Now, we can plug this value of x_1 in the first equation to obtain $-14x_2 + \sqrt{3}x_2 = 2$. This means that $x_2 = \frac{2}{\sqrt{3}-14}$. Now we can go back to the initial equation $x_1 = -7x_2$ and plug the value of x_2 to obtain the value of x_1: $x_1 = \frac{-14}{\sqrt{3}-14}$. Since we have found the values of x_1 and x_2 such that both equatios simultaneously hold, we have solved the system.

Elementary operations on systems of equations. There are three elementary operations that can be performed on the equations of a system of equations, or equivalently on the corresponding augmented matrix.

- We can replace any equation by the sum of itself and a multiple of another equation.
- We can exchange the order of two equations.
- We can multiply an equation by a nonzero multiple of itself.

For instance, consider the system

$$\begin{cases} 2x_1 + \sqrt{3}x_2 = 2 \\ x_1 + 7x_2 = 0. \end{cases}$$

We can replace the first equation by the sum of itself and -2 times the second equation. This looks like $2x_1 + \sqrt{3}x_2 - 2x_1 - 14x_2 = 2$ which gives the new equation $\sqrt{3}x_2 - 14x_2 = 2$. Compare the procedure here and Example 2.3. Does it seem familiar? Of course, exchanging equations means simply flipping the order they appear in the system. In Example 2.3 we did something like that, implicitly, when we started from the second equation rather than the first one. The third operation simply says that between taking, say, the second equation or a multiple by 2 of it, it does not change anothing. In fact, $x_1 + 7x_2 = 0$ if and only if $2x_1 + 14x_2 = 0$. The same operations can be applied directly to the augmented matrix corresponding to the system, since it contains all the coefficients of the system.

Exercise 2.4. Translate the solution given in Example 2.3 in terms of matrix operations.

When performing the operations to the rows of an augmented matrix, we say that we are performing elementary row operations on the matrix. Two systems whose augmented matrices are related by elementary row operations are said to be *row equivalent*. If two systems have row equivalent augmented matrices, then their solutions sets are the same. This gives us the idea that to solve a system we can take the augmented matrix and perform row operations until we obtain some very easy matrix whose system is very easy to solve.

Now the question arises. What is an augmented matrix whose corresponding system is easy to solve? How does it look like? Well, we have already seen an instance of this. Let us do another computation to make it clearer.

Example 2.5. We want to determine whether the following system is consistent (i.e. it admits at least one solution):

$$\begin{cases} x_1 + x_2 - 3x_3 = 0 \\ -x_2 + 2x_3 = 3 \\ 2x_1 + 5x_3 = 1. \end{cases}$$

The corresponding augmented matrix is given by

$$\begin{bmatrix} 1 & 1 & -3 & 0 \\ 0 & -1 & 2 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}.$$

Now, let us sum -2 times the first row to the last one. We get the new matrix

$$\begin{bmatrix} 1 & 1 & -3 & 0 \\ 0 & -1 & 2 & 3 \\ 0 & -2 & 11 & 1 \end{bmatrix}.$$

Now, we can eliminate the -2 term in the last row by adding -2 times the second row to the third one to get

$$\begin{bmatrix} 1 & 1 & -3 & 0 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 7 & -5 \end{bmatrix}.$$

Now, I claim that the system corresponding to such an augmented matrix is simple to solve. In fact, this is the system

$$\begin{cases} x_1 + x_2 - 3x_3 = 0 \\ -x_2 + 2x_3 = 3 \\ 7x_3 = -5. \end{cases}$$

and we can immediately find the value of $x_3 = -5/7$. We can then plug this in the other two equations to find

$$\begin{cases} x_1 + x_2 - 3(-5/7) = 0 \\ -x_2 + 2(-5/7) = 3 \\ x_3 = -5/7. \end{cases}$$

from which we get $x_2 = -3 - 10/7 = 31/7$. Now we know both x_2 and x_3, and we can plug them in the first equation to get x_1.

So, whe have found that the system is consistent, and that the number of solutions is exactly 1.

2.3. **Echelon form.** Linear systems can get out of hand very rapidly in applications. They can become very large, consisting very often of hundreds of thousands or millions of equations and variables. Of course, this requires a more structured and algorithmical approach to solving these systems. While the discussion up to know gives us a way of solving linear systems, it is still rather inefficient. The *Echelon* form of a matrix allows us to produce an algorithm that is very effective in solving very large systems of linear equations.

Before showing how to algorithmically obtain the echelon form of a matrix, we define what being "echelon" means to start with.

Definition 2.6. A matrix is said to be in echelon form (more precisely row echelon form) if the following properties hold:

- All rows consisting of all zeros are below rows having nonzero terms.
- Each leading entry of a row (i.e. the first nonzero term in a row) is in a column to the right of the leading entry of the row above it.

Morevoer, a matrix is said to be in reduced echelon form (more precisely reduced row echelon form) if it satisfies the additional properties:

- The leading entry in each nonzero row is 1.
- Each leading 1 is the only nonzero entry in its column.

Example 2.7. The following matrix is in echelon form:

$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 3 & 1 \\ 0 & 0 & 0. \end{bmatrix}$$

The following matrix is in reduced echelon form:

$$\begin{bmatrix} 1 & 0 & \sqrt{3} \\ 0 & 1 & 2 \\ 0 & 0 & 0. \end{bmatrix}$$

Theorem 2.8. *Any matrix A is row equivalent to a matrix in reduced echelon form. Moreover, this matrix is unique.*

When U is a matrix in (reduced) echelon form that is row equivalent to A, we say that U is a (reduced) echelon form of A.

Algorithm for finding the echelon form. We now want to describe an algorithm to find a reduced echelon form of A through row operations.

Definition 2.9. A pivot position in the matrix A is the location of a leading 1 in the reduced echelon form of A.

We show the procedure through an example, for simplicity.

Example 2.10. Consider the matrix

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 1 & 5 \\ 3 & 2 & 1 \end{bmatrix}$$

and find a reduced echelon form for it.

First of all, select the first nontrivial column. This is the first column. Then, we need to have zeros below nonzero entries. So, we switch the third row and the first one to have a leading coefficient on top that is nonzero. We get

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 5 \\ 0 & 3 & 1 \end{bmatrix}$$

Now, we need to create zeros below the leading coefficient of the first row. This means that we can subtract to the second row $-2/3$ times the first row. This gives

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1-4/3 & 5-2/3 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & -1/3 & 13/3 \\ 0 & 3 & 1 \end{bmatrix}.$$

At this point, to obtain an echelon form, we need to perform another elementary row operation and eliminate the leading term of the third row. To do this, we cannot use the first row, because it would otherwise re-introduce a leading term in the first column. We subtract to the third row, 9 times the second row. We get

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & -1/3 & 13/3 \\ 0 & 0 & 39 \end{bmatrix}.$$

This is an echelon form for A. Now, we want to turn this echelon form into a reduced echelon form. We take the rightmost pivot, which here is 39, and turn any element above it into a zero. To do this, we just need to add to the second row, $1/9$ times the third row. This would give us

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & -1/3 & 0 \\ 0 & 0 & 39 \end{bmatrix}.$$

Similarly, we subtract $1/39$ times the third row to the first one. And we get

$$\begin{bmatrix} 3 & 2 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 39 \end{bmatrix}.$$

To complete, we take the rightmost pivot position that has nontrivial entries above. This is $-1/3$. We want to cancel the term above it. To do so, we multiply the second row by 6, and add it to the first row to get

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 39 \end{bmatrix}.$$

This is our reduced echelon form.

The algorithm now, more generally, is the following.

1 First, individuate the first column that has nontrivial elements. This is going to be the first pivot.
2 Position a row whose leading coefficient lies in this column to the top of the matrix, using the row exchange.
3 Use multiples of this row to annihilate all elements below the leading coefficient of it.
4 Repeat steps 1 to 3 to the submatrix obtained by considering only the rows below the first one, and the columns on the right of the first nontrivial column.
5 At some point, the previous steps arrive at a point where there are no more operations needed, and we have an echelon form.
6 Now, select the rightmost pivot, and sum multiples of this row to all the rows above that have nontrivial elements above the pivot so that we get all zeros above it.
7 Repeat this step for the pivots moving leftward.
8 The algorithm stops at some point giving a reduced echelon form.

Solving linear systems whose augmented matrix is in reduced echelon form. Suppose we have a system whose augmented matrix is in reduced echelon form, through the previous algorithm. For example, this could be the matrix

$$\tilde{A} = \begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which corresponds to the linear system

$$\begin{cases} x_1 - 5x_3 = 1 \\ x_2 + x_3 = 4 \ . \end{cases}$$

We can solve the system now simply by writing x_1 and x_2 in terms of x_3. We have

$$\begin{cases} x_1 = 5x_3 + 1 \\ x_2 = -x_3 + 4 \ . \end{cases}$$

Observe that there are no restrictions on x_3 and in this case we say that x_3 is a free variable. In fact, x_3 can take any value, and this would force x_1 and x_2 to have a specific numerical value determined through the equations above. This system is consistent, and it has infinitely many solutions.

One more computation.

Example 2.11. Consider the system of linear equations having augmented matrix

$$\tilde{A} = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -2 & -1 & 1 \end{bmatrix}$$

First, we swap second and third rows.

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & -2 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Then, we cancel all elements above the element 1 in the third row, by summing the third row to the second, and subtracting twice the third row to the first. We get

$$\begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Then we do the same to cancel all elements above the -2 in the second row. We get

$$\begin{bmatrix} 1 & 0 & 0 & 5/2 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

This gives us the solution $x_1 = 5/2$, $x_2 = -3/2$ and $x_3 = 2$.

Definition 2.12. For the augmented matrix of a linear system of equations in echelon form, we say that a variable is free, if it does not correspond to a pivot position.

Example 2.13. Consider the matrix in reduced echelon form

$$\begin{bmatrix} 1 & 0 & 2 & 7 \\ 0 & 1 & 5 & \sqrt{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, the variable x_1 corresponds to the top left 1, which is pivot, and x_2 corresponds to the leading coefficients of the second row, which is pivot as well. However, x_3 does not correspond to any pivot position, and it is therefore free.

Theorem 2.14. *A linear system is consistent if and only if in the echelon form of the augmented matrix there is no row of type*

$$[0 \ \cdots \ 0 \ b].$$

If the linear system is consistent, then the solution is either unique (obtained by the substitution procedure shown above), or it has infinitely many solutions when it has free variables.

3. Vector equations through matrices

A *vector* is a matrix that consists of a single column. When we have a matrix consisting of a single row, we will say that this is a *row vector* and say that this is the *transpose* of a vector. Equality of vectors means that **all** the entries of each vector (in their respective positions) are the same. For instance, the two vectors

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

are equal, but the two vectors

$$\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

are not equal.

Vectors can be added together (componentwise) and subtracted (componentwise) as

$$\begin{bmatrix}1\\3\end{bmatrix}+\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix}.$$

Also, we can multiply a vector $\mathbf{v}$ by a scalar (i.e. a number c) simply by multiplying all entries of $\mathbf{v}$ by c, as in the following example

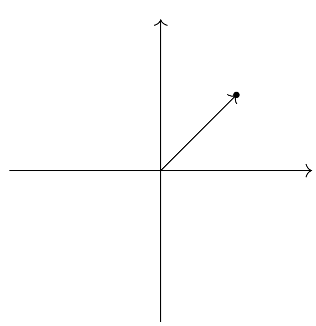
$$2\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}.$$

The use of vectors in terms of linear systems is the following. Consider a linear system whose associated matrix is A, and whose augmented matrix $\tilde{A}$ is obtained by adding a column consisting of $b_1,\dots,b_n$. Then, we can write the system as

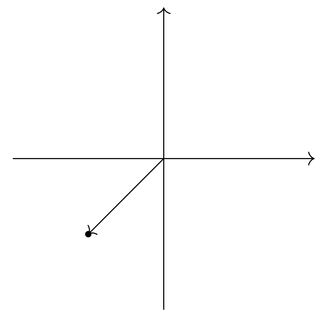
$$A\mathbf{x}=\mathbf{b},$$

where $\mathbf{x}=\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}$ is the vector of indeterminates (the variables), and $\mathbf{b}=\begin{bmatrix}b_1\\\vdots\\b_n\end{bmatrix}$ is the vector containing all the entries of the column that we have to add to A to obtain $\tilde{A}$. Here, the product between A and $\mathbf{x}$ has the meaning of a product of a matrix by a vector (a special case of a product of a matrix by a matrix), and is defined as follows. The output of it is a vector, and the first entry of the vector $A\mathbf{x}$ is given by the sum of the product of the terms of the first row of A with the elements of $\mathbf{x}$ (term by term). This is given by $A_{11}x_1+A_{12}x_2+\cdots+A_{1n}x_n$. More generally, the entry of $A\mathbf{x}$ in position i is given by multiplying the i^{th} row of A by the elements of $\mathbf{x}$ and summing them up, as $A_{i1}x_1+A_{i2}x_2+\cdots+A_{in}x_n$.

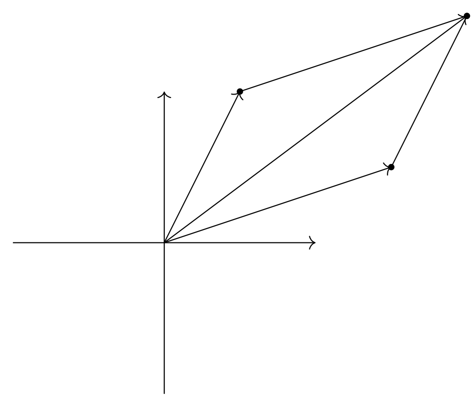
The set of vectors of type $\mathbf{x}=\begin{bmatrix}a_1\\\vdots\\a_n\end{bmatrix}$, consisting of a column of n numbers, is denoted by $\mathbb{R}^n$. The space $\mathbb{R}^2$, in particular, is the set of vectors consisting of a pair of numbers $\mathbf{x}=\begin{bmatrix}a_1\\a_2\end{bmatrix}$. Such vectors correspond to the points in the plane. For instance, $\mathbf{x}=\begin{bmatrix}1\\\vdots\\1\end{bmatrix}$ is shown in the plane as



where the arrow goes from the point $(0,0)$ to the point $(1,1)$. Similarly, the vector $\mathbf{x} = \begin{bmatrix} -1 \\ \vdots \\ -1 \end{bmatrix}$ is shown in the plane as



The rules for summing and subtracting vectors take a geometric form on the plane, as in the following case



which is in general called the *parallelogram rule*. To subtract two vectors, one takes the negative of a vector, and then sum them following the previous graphical rule.

When dealing with $\mathbb{R}^3$, one can follow the same rules, but now the cartesian system is going to be 3-dimensional. With higher dimensions it is more complicated to represent the vectors.

Question 3.1. What is the graphical depiction of the operation of multiplying a vector by a scalar? Hint: take a vector in the plane, multiply it by 2, and draw what you obtain.

A linear system of equations can also be compactly represented as a vector equation using the form

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}.$$

The meaning of this equation is that the variables (as x_1) multiply all the entries in the vector (the rule of multiplying a vector by a scalar), and this gives a column as in a linear system of equation. At the end, on the RHS of the equality, we have all the coefficients $\mathbf{b}$.

Definition 3.2. For p vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ in $\mathbb{R}^n$, and p coefficients $c_1, \dots, c_p$, we call the quantity $c_1\mathbf{v}_1, \dots, c_p\mathbf{v}_p$ a *linear combination* of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$. The set of linear combination of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ is called the *linear span*, and it is denoted by the symbol

$$\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}.$$

In view of the previous definition, it follows that a vector equation of type

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has a solution if and only if the vector $\mathbf{b}$ lies in the span of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$, where $a_1, \ldots, a_n$ are vectors in some $\mathbb{R}^k$. Note that n was indicating the dimension of $\mathbb{R}^n$ before, but now it is the number of vectors. This should not cause any confusion.

Geometric meaning of the span of two vectors. Before considering the case of two vectors, let us fix a single vector $\mathbf{v}$, and let us take the span of $\mathbf{v}$. We want to describe this. Linear combinations consist of multiplying vectors by some coefficeints, and then summing them together. Since there is just a single vector here (which is $\mathbf{v}$), this means that the only operation we can do is to take multiples of it. Therefore, the span of $\mathbf{v}$ consists of all vectors obtained as $a\mathbf{v}$ where a is some number.

What we have just described is the geometric notion of a line! This is because fixed $\mathbf{v}$, multiplying it by scalars, we can only get all the points in the same direction, without ever getting out of this direction. This relates to Question 3.1, where we have seen that multiplying by a scalar returns the "same arrow" (vector), but with longer or shorter size.

Now, when we consider the span of two vectors, we are considering all the vectors that can be written as $a\mathbf{v} + b\mathbf{w}$. If $\mathbf{w}$ is proportional to $\mathbf{v}$, i.e. we have $\mathbf{w} = k\mathbf{v}$ for some number $k \neq 0$, then $\mathbf{w}$ is in the space of $\mathbf{v}$, and the span of $\mathbf{v}$ and $\mathbf{w}$ is the same as the span of $\mathbf{v}$, so we are back to the previous situation. When $\mathbf{v}$ and $\mathbf{w}$ are not proportional, all their linear combinations give us vectors that lie on the same plane as $\mathbf{v}$ and $\mathbf{w}$. Therefore, the span of $\mathbf{v}$ and $\mathbf{w}$ is the plane passing through $\mathbf{v}$ and $\mathbf{w}$.

3.1. **System of equations, and matrix equations.** We have seen above that we can write a system of equations as a matrix equation of type $A\mathbf{x} = \mathbf{b}$, where A is the matrix containing all the coefficients, $\mathbf{x}$ is the vector of variables, and $\mathbf{b}$ is the vector of coefficients on the right hand side of the equalities of the system. We want to use this equation to determine when the system has solutions, and to obtain them.

Proposition 3.3. *The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.*

Theorem 3.4. *Let A be an $m \times n$ matrix, and consider the matrix equation $A\mathbf{x} = \mathbf{b}$. The following conditions are equivalent.*

- *For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution;*
- *Each $\mathbf{b}$ in $\mathbf{R}^m$ is a linear combination of the columns of A;*
- *The columns of A span $\mathbf{R}^m$;*
- *A has a pivot in every row.*

Solution sets of linear systems of equations. System of equations where $\mathbf{b} = 0$, i.e. that can be written in the form $A\mathbf{x} = \mathbf{0}$, are called *homogeneous systems*. There is a simple characterization of the existence of solutions for such linear systems or, equivalently, the associated matrix equation.

Proposition 3.5. *The homogeneous equation $A\mathbf{x} = \mathbf{b}$ has a nontrivial solution if and only if the equation has at least one free variable.*

Example 3.6. Consider the system of equations

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 0 \\ -3x_1 - 2x_2 + 4x_3 = 0 \\ 6x_1 + x_2 - 8x_3 = 0 \end{cases}$$

Using the algorithm for finding a reduced echelon form, we see that the system (or better its augmented matrix) is row equivalent to

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives us the system (equivalent to the one we started with)

$$\begin{cases} x_1 - \frac{4}{3}x_3 = 0 \\ x_2 = 0 \\ 0 = 0 \end{cases}$$

Since $-\frac{4}{3}$ is a nontrivial entry that is not in a pivot position, it is going to be a free variable. This means by Proposition 3.5 that the system has a nontrivial solution.

In fact, we can find the whole solution set. From the second equation we have $x_2 = 0$. From the first equation we have that $x_1 = \frac{4}{3}x_3$. This means that whenever we pick a value for x_3, we automatically find the value for x_1 such that the equation holds. This is because x_3 is free. We can give any value to x_3. The values of x_1 and x_2 are not free, but they are given by $\frac{4}{3}x_3$ and 0 respectively. The solution set is given by all triples of numbers $(\frac{4}{3}x_3, 0, x_3)$, where x_3 is an arbitrary number in $\mathbb{R}$.

Let us now consider *nonhomogeneous systems*, which are systems where $\mathbf{b} \neq \mathbf{0}$. The main idea is that of finding solutions of the corresponding homogeneous equation (by discarding the $\mathbf{b}$ term), and then obtaining all solutions for the original nonhomogeneous equation by translating the homogeneous solution set.

Example 3.7. Consider the equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$

and $\mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$ We can see that the reduced echelon form of the augmented matrix $\tilde{A}$ is given by

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives solution $x_1 = -1 + \frac{4}{3}x_3$ and $x_2 = 2$, with x_3 free variable. One can see that the general solution can be written in the form of

$$\mathbf{x} = \mathbf{p} + x_3\mathbf{v},$$

where $\mathbf{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$. This shows that to know all the solutions of the equation, we have to know the vector $\mathbf{v}$, take all its multiples, and then translate it by $\mathbf{p}$. It turns out that $\mathbf{v}$ is a solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. So, from the homogeneous solutions, upon translating by a vector we obtain all solutions of the nonhomogeneous equation.

The result from the previous example is actually true in general.

Theorem 3.8. *Suppose that* $A\mathbf{x} = \mathbf{b}$ *has a solution (for a given* $\mathbf{b}$*), and let* $\mathbf{p}$ *be such a solution. Then, all the solutions to the equation are obtained as* $\mathbf{p} + \mathbf{w}$*, where* $\mathbf{w}$ *is a solution of the homogeneous equation (*$A\mathbf{x} = \mathbf{0}$*).*

3.2. **Applications.** We consider now an application of the study of linear systems of equations. Namely, we consider how to balance a chemical reaction. Chemical reactions generally have reactants that very over time and the dynamics of such variation is of great interest in chemistry. However, reactions very often tend to reach an equilibrium point. We will see that such a situation can be studied with the tools we have learned so far.

The reaction obtained by burning propane is given by

$$x_1 \cdot \mathrm{C_3H_8} + x_2 \cdot \mathrm{O_2} \longrightarrow x_3 \cdot \mathrm{CO_2} + x_4 \cdot \mathrm{H_2O}.$$

This gives us a three dimensional space, where we have vectors $\begin{bmatrix} c \\ h \\ o \end{bmatrix}$ where the first entry is the number of carbon (C) atoms, the second entry is the number of hydrogen (H) atoms, and the last entry is the number of oxygen (O) atoms.

Balancing the reaction means that the atoms corresponding to each type (i.e. Carbon, Hydrogen and Oxygen), need to be the same before and after the reaction (Lavoisier's principle of mass conservation). In other words, we have to get a linear equation for the vectors of type $\begin{bmatrix} c \\ h \\ o \end{bmatrix}$ where x_1, x_2, x_3 and x_4 are such that the number of atoms are the same on both sides of the arrow in the reaction. We get the equality

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Now, this equation can be turned into a system of equations which is

$$\begin{cases} 3x_1 - x_3 = 0 \\ 8x_1 - 2x_4 = 0 \\ 2x_2 - 2x_3 - x_4 = 0. \end{cases}$$

Solving this system (as we have done so far) gives us the solution for the balanced reaction.

3.3. **Linear Independence.** We have seen that the span of vectors is a useful notion to describe the solution set of some matrix equations, and therefore of their associated systems. However, it can happen that given certain vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, the information that some of the vectors carry is redundant. For instance, consider the simple case of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

It is clear that we can write $\mathbf{v}_2 = 2\mathbf{v}_1$. So, whenever we get a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ we have $a\mathbf{v}_1 + b\mathbf{v}_2 = 1\mathbf{v}_1 + 2b\mathbf{v}_1 = (1 + 2b)\mathbf{v}_1$. So, the span of $\mathbf{v}_1$ and the span of $\mathbf{v}_1$ and $\mathbf{v}_2$ is the same:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{v}_1\}.$$

This concept of redundance among vectors, or lack of it, is formalized in the notion of linear of *linear independence.*

Definition 3.9. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be vectors in $\mathbb{R}^n$. They are said to *linearly independent* if the equation

$$x_1\mathbf{v}_1 + \cdots + x_m\mathbf{v}_m = 0,$$

admits only the trivial solution $x_1 = \ldots = x_m = 0$. If there is a nontrivial solution (i.e. with at least one of the x_i nonzero) to the previous equation, then the set of vectors is said to be *linearly dependent.*

In the example above, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent. In fact, it can be seen that $2\mathbf{v}_1 - \mathbf{v}_2 = 0$, meaning that there exists a nontrivial solution to a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, where $x_1 = 2$ and $x_2 = -1$. Being linearly dependent, their span can be reduced to the span of just (either) one of them.

Our question now is how to check whether certain vectors are linearly dependent or independent. Let us show the idea through an example.

Example 3.10. Consider the three vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$. We would like to understand whether these vectors are linearly independent or not. In other words, we should verify whether there exist coefficients x_1, x_2 and x_3 (not all of them zero) such that

$$x_1\mathbf{v}_1 + x_2 + \mathbf{v}_2 + x_3\mathbf{v}_3 = 0.$$

In other words, our problem has become the same as finding nontrivial solutions of a linear system. We can see that taking $x_1 = 3$, $x_2 = 3$ and $x_3 = -1$ we solve the equation above, showing that there are nonzero coefficients such that a linear combination of the $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ is zero. This means that the vectors are linearly dependent.

There are some very useful criteria for checking whether sets of vectors are linearly dependent or not. Here is a result that immediately tell us that some vectors are linearly dependent.

Proposition 3.11. *Let $\{v_1, \ldots, v_p\}$ be a set of vectors in $\mathbb{R}^n$. If either of the two conditions is satisfied, then the set is linearly dependent:*

- *One of the vectors is the zero vector* $\mathbf{0}$*;*
- *The number of vectors p is larger than the dimension n of the space $\mathbb{R}^n$.*

Proof. Suppose that the first condition holds, and that one of the vectors is zero. Without loss of generality, suppose that this vector is $\mathbf{v}_1 = \mathbf{0}$. Then, choosing $x_1 = 1$, and all the $x_2, \ldots, x_p$ to be zero we find a nontrivial linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ that gives zero. The vectors are therefore linearly dependent.

Consider now the second case, i.e. suppose that $p > n$. Then, the equation

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0},$$

corresponds to a system of p equations in n variables. So, the nunmber of variables is larger than the number of equations, meaning that there will be free variables. This implies that there exists a nontrivial solution, meaning that the vectors are linearly dependent. □

We also have the following.

Theorem 3.12. *A set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of vectors in $\mathbb{R}^n$ is linearly dependent if and only if at least one vector is a linear combination of the other vectors.*

Proof. Suppose that the set S is linearly dependent, and assume that $\mathbf{v}_1 \neq 0$ (observe that if $\mathbf{v}_1 = \mathbf{0}$ then it is clearly a linear combination of the other vectors). This means that there exist coefficients (not all trivial) $x_1, \dots, x_p$ such that

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}.$$

Since the x coefficients are not all trivial, it means that we can find one of them (say x_1 for simplicity and clarity) which is not zero. This means that

$$\mathbf{v}_1 = -\frac{x_2}{x_1}\mathbf{v}_2 - \cdots - \frac{x_p}{x_1}\mathbf{v}_p.$$

This is the definition of $\mathbf{v}_1$ being a linear combination of the other vectors, and we have proved that if S is linearly dependent, then one of the vectors can be written as linear combination of the other vectors.

Let us do the opposite implication. Suppose that we can write one of the vectors (let us say $\mathbf{v}_1$ again for simplicity and clarity) as linear combination of the other vectors. If $\mathbf{v}_1 = \mathbf{0}$ then we are done from the Proposition above. So, take $\mathbf{v}_1 \neq \mathbf{0}$. This means that we have some equation of type

$$\mathbf{v}_1 = x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p,$$

where the x coefficients are not all trivial. Then we have a nontrivial linear combination

$$\mathbf{v}_1 - x_2\mathbf{v}_2 - \cdots - x_p\mathbf{v}_p = \mathbf{0}.$$

This completes the proof. □

3.4. **Linear Maps.** Linear maps (or linear transformations) are functions $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ satisfying the following property:

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y}),$$

for every choice of vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and any choice of numbers α and β.

Equivalently, when checking whether a map T is linear or not, we can verify the two conditions

- $T(\alpha\mathbf{x}) = \alpha T(\mathbf{x})$
- $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$.

Example 3.13. The function $f(x) = 5x$ is a linear map $f : \mathbb{R} \longrightarrow \mathbb{R}$. On the contrary, the function $f(x) = \sin(x)$ is not linear, since $\sin(x + y) \neq \sin(x) + \sin(y)$.

The following example shows the class of linear maps whose study is one of the main interests in linear algebra.

Example 3.14. Any $n \times m$ matrix A gives rise to a linear map $\mathbb{R}^m \longrightarrow \mathbb{R}^n$. The map is obtained by using the matrix-vector product rule. For example, consider the 3×2 matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ -7 & 0 \end{bmatrix}.$$

Then, whenever we get a vector in $\mathbb{R}^2$, which is an object of type $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we can perform the matrix-vector multiplication to obtain:

$$A\mathbf{x} = \begin{bmatrix} 2x_1 + x_2 \\ 2x_2 \\ -7x_1 \end{bmatrix},$$

which gives another vector. In other words, we start with a vector in $\mathbb{R}^2$, and we get a vector in $\mathbb{R}^3$. This is a map $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$. We still need to understand whether this is linear or not. Consider a scalar multiple of $\mathbf{x}$, say $\alpha\mathbf{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}$. Then we get

$$A(\alpha\mathbf{x}) = \begin{bmatrix} 2\alpha x_1 + \alpha x_2 \\ 2\alpha x_2 \\ -7\alpha x_1 \end{bmatrix} = \alpha \begin{bmatrix} 2x_1 + x_2 \\ 2x_2 \\ -7x_1 \end{bmatrix} = \alpha A(\mathbf{x}).$$

The first property is satisfied. To verify the second, consider now $\mathbf{x}$ as before, and also take $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. We have

$$A(\mathbf{x}+\mathbf{y}) = \begin{bmatrix} 2(x_1 + y_1) + x_2 + y_2 \\ 2(x_2 + y_2) \\ -7(x_1 + y_1) \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ 2x_2 \\ -7x_1 \end{bmatrix} + \begin{bmatrix} 2y_1 + y_2 \\ 2y_2 \\ -7y_1 \end{bmatrix} = A(\mathbf{x}) + A(\mathbf{y}).$$

So, starting from a matrix, we obtain a linear map automatically. We will show, shortly, that all linear maps $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ arise in this way.

Example 3.15. A very important class of linear maps, whose applications are found throughout functional analysis and numerical analysis (although in the infinite dimensional case!), is the class of projections. Roughly speaking, these maps take general vectors and return the components of the vectors that lie in some chosen subspace. For instance, consider the three dimensional space $\mathbb{R}^3$, and say that we want to project vectors from $\mathbb{R}^3$ onto the plane $\mathbb{R}^2$ determined by the x and y coordinates (i.e. the first two coordinates). This map, which we call P_3 takes $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and gives back the vector $P_3\mathbf{v} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$, where the third component has been removed! This linear map can be written in the form of a matrix multiplying the input vectors in a very simple way. This is

$$P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A direct inspection (left as an exercise to the reader!) shows that matrix-vector multiplication of the previous matrix and $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ gives exactly $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ as claimed.

A fundamental property of linear maps is the following.

Proposition 3.16. *Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ denote a linear map. Then, we have*

$$T(\mathbf{0}) = \mathbf{0}.$$

Proof. We have

$$T(\mathbf{0}) = T(\mathbf{x} - \mathbf{x}) = T(\mathbf{x}) - T(\mathbf{x}) = \mathbf{0},$$

for any choice of $\mathbf{x}$.

Alternatively, another proof is obtained by considering

$$T(\mathbf{0}) = T(0 \cdot \mathbf{x}) = 0 \cdot T(\mathbf{x}) = \mathbf{0},$$

again for any choice of $\mathbf{x}$. □

The matrix of a linear map. First observe that any vector in $\mathbb{R}^n$ can be written as a linear combination of the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ defined by

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots \quad , \mathbf{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

where $\mathbf{e}_i$ consists of zeroes everywhere, but at the i^{th} row, where there is 1. It is also simple to see that the way of writing any vector as a linear combination of such vectors is unique. We will later see that this is an instance of the notion of *basis* of a vector space.

Proposition 3.17. *Any linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the linear map associated to a unique matrix A. In other whords*

$$T(\mathbf{x}) = A\mathbf{x}.$$

We show the result through an example. The generalization of it being straightforward.

Example 3.18. Consider the linear map $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 3x_1 + x_2 \\ 2x_2 \end{bmatrix}$. A direct verification shows that this is indeed a linear map. Since we can decompose the vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ into the sum of the two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ multiplied by x_1 and x_2, respectively, we can obtain the value of T on $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ by looking at the value of T on $\mathbf{e}_1$ and $\mathbf{e}_2$. We have that $T(\mathbf{e}_1) = 3\mathbf{e}_1$, and $T(\mathbf{e}_2) = \mathbf{e}_1 + \mathbf{2}e_2$. Now, we can construct a matrix A where we place the coefficients of $\mathbf{e}_1$ and $\mathbf{e}_2$ from the previous computation in the columns. The first column being $T(\mathbf{e}_1)$, and the second column corresponding to $T(\mathbf{e}_2)$. The first column will have only a 3 on top, because $T(\mathbf{e}_1)$ does not have a $\mathbf{e}_2$ component. The second column will have 1 on top (since $\mathbf{e}_1$ appears in $T(\mathbf{e}_2)$ without coefficients multiplying it), and a 2 corresponding to the $\mathbf{e}_2$ term in $T(\mathbf{e}_2)$. We get

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}.$$

A direct verification shows that $A \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ gives the same output as T.

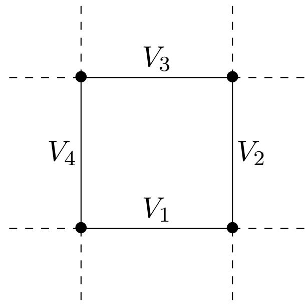


FIGURE 1. Loop in a circuit to which Krichhoff's second Law applies

3.5. **Application to electrical circuits.** Consider an electrical circuit where electricity is flowing (i.e. the electrons are moving), and suppose that a resistor (such as a light bulb) is part of the circuit. Due to the presence of something that uses the electricity, there is a drop in the voltage (unit of meaasure V for Volts). A battery provides voltage so that the current keeps flowing. The resistance of the resistor is measured in Ohms, whose unit symbol is Ω, and the current is measured in Ampéres, denoted by A. At a node, i.e. a junction of the circuit of multiple wires, the currents of each wire take a sign indicating whether the flow is incoming $(+)$, or outgoing $(-)$. The voltage drop due to a light bulb is given by

$$R = VI, \tag{2}$$

where R is the resistance, V the voltage and I the current.

The first Kirchhoff's Law states that

- At each node, there algebraic sum (with signs) of the currents needs to be zero. In other words, wathever comes in a node, must also go out.

The second Kirchhoff's Law states that

- The directed sum of the potential differences (voltages) around any closed loop is zero. Here the sources have plus sign if the current agrees with the orientation of the source, and they are negative otherwise. See Figure 1.

Consider now a circuit as in Figure 2.

The meaning of the circuit is as follows. Each loop has a current of I_1, I_2 or I_3, and the direction is assumed to be counterclockwise in all loops. There are three Voltage generators denoted by a voltage V, where the top voltage is taken with $+$ because the battery has the same orientation of the current, while in the other two cases we have the batteries in the opposite direction. The segments have resistances given by some Ω. In this circuit, the currents I_1, I_2 and I_3 are not known, and we want to use the Kirchhoff's second law to calculate them.

In the first loop (top), we have the current I_1 going through three resistors, corresponding to a drop of

$$4I_1 + 4I_1 + 3I_1 = 11I_1.$$

However, this is not all that needs to be considered. In fact, the lower loop shares a wire with the first loop. Therefore, we need to consider the voltage corresponding to it and add/subctract it to $11I_1$. The current in the second loop (middle) flows in the opposite direction than the one in the first loop, on the segment that they share. This means that we need to subtract $3I_2$ from $11I_1$. For

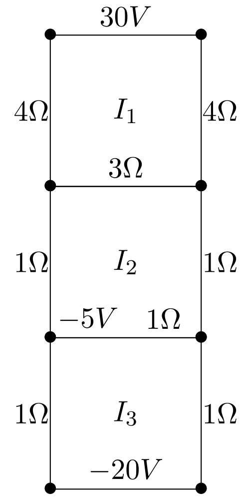


FIGURE 2. Circuit having three loops

the first loop, we also have a generator of $30V$. Therefore, Kirchhoff's law gives us an equality

$$11I_1 - 3I_2 = 30.$$

Similar considerations for the voltages in the second loop give

$$-3I_1 + 6I_2 - I_3 = 5,$$

and for the third one

$$-I_2 + 3I_3 = -25.$$

We can now obtain a system of three equations whose solution gives the three unknown currents.

$$\begin{cases} 11I_1 - 3I_2 = 30 \\ -3I_1 + 6I_2 - I_3 = 5 \\ -I_2 + 3I_3 = -25. \end{cases}$$

Solving this system gives a solution of $I_1 = 3A$, $I_2 = 1A$ and $I_3 = -8A$.

4. Matrix Algebra

In this section we focus on the set of all matrices over $\mathbb{R}$ (or $\mathbb{C}$) with a fixed size, and consider several algebraic operations on this set. Our focus will be on real numbers, as before. We will denote the set of $m \times n$ matrices with entries in $\mathbb{R}$ with the symbol $\mathrm{M}_{\mathbb{R}}(m, n)$, and a similar notation holds when the entries are in $\mathbb{C}$.

There are two basic operations of matrices that directly generalize scalar multiplication and sum of vectors.

Definition 4.1. Let r be a number, and let A and B be $m \times n$ matrices (i.e. elements of $\mathrm{M}_{\mathbb{R}}(m, n)$). Then, we define the new matrix $r \cdot A$ by multiplying all the entries of A by the number r, and we define the matrix $A + B$ by summing all the entries of A and B in their respective positions.

Let us now see the definition with an example.

Example 4.2. Let $r = 3$, $A = \begin{bmatrix} 1 & 0 & 7 \\ -3 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 & 1 \\ -7 & 2 & -1 \end{bmatrix}$. Observe that A and B are 2×3 matrices. We have

$$rA = 3A = A = \begin{bmatrix} 3 & 0 & 21 \\ -9 & 6 & -3 \end{bmatrix}$$

and

$$A + B = \begin{bmatrix} 1 & 0 & 7 \\ -3 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -3 & 1 \\ -7 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 8 \\ -10 & 4 & -2 \end{bmatrix}.$$

The following result is easy to prove dorectly from the definitions, and it is left to the reader as an exercise.

Proposition 4.3. *Let A, B, C be $m \times n$ matrices, and let r, s be numbers. Also, denote by $\mathbb{0}$ the $m \times n$ matrix consisting of all zero entries. Then, the following equalities hold:*

- $A + B = B + A$;
- $(A + B) + C = A + (B + C)$;
- $A + \mathbb{0} = A$;
- $r(A + B) = rA + rB$;
- $(r + s)A = rA + sA$;
- $r(sA) = (rs)A$.

4.1. **Matrix multiplication.** We consider now an extremely important operation of matrices. Namely, we introduce the multiplication of two matrices. The reason for introducing this operation is that we want to represent the composition of linear maps in their matrix form. We know that whenever we have a linear map, this can be written as a matrix. Now, we can compose linear maps (they are functions!). Our question is how do we describe the matrix corresponding to the composition of linear maps?

The answer is the product of matrices. We will derive the matrix of a composition, and we will define the product to be exactly this matrix.

Let T_1 and T_2 be two linear maps. Assume that $T_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. So, T_1 takes as input a vector of n entries, and returns a vector of m entries. We know that the matrix for T_1 is an $m \times n$ matrix A_1. Same thing with T_2. If $T_1 : \mathbb{R}^m \longrightarrow \mathbb{R}^k$, then its corresponding matrix A_2 is a $\times k \times m$ matrix. Observe that we can perform the composition of linear maps $T_2 \circ T_1$ (why?).

We want to obtain the matrix for $T_2 \circ T_1$ from A_1 and A_2.

Example 4.4. We will follow the procedure to obtain the maatrix for $T_2 \circ T_1$ in the case of three dimensional vectors, for simplicity. The general case is exactly the same, but with more indices. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be 3-dimensional vector. Let A_1 be the matrix associated with the linear map T_1, and let A_2 be the matrix associated with T_2. Let us now compute $T_2(T_1(\mathbf{x}))$ by multiplying A_1 and $\mathbf{x}$, and then multiplying the result by A_2. For $A_1 \cdot \mathbf{x}$, we have the equality

$$A_1 \cdot \mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3,$$

where $\mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_3$ are the columns of A_1 (the columns can be seen as vectors!). We leave it to the reader to verify the previous equality. Now, when we apply A_2 on $A_1 \cdot \mathbf{x}$, due to the linearity of matrix-vector multiplication (the matrix represents a linear map!), we find that

$$A_2 \cdot (A_1 \cdot \mathbf{x}) = x_1 A_2(\mathbf{c}_1) + x_2 A_2(\mathbf{c}_2) + x_3 A_2(\mathbf{c}_3).$$

But this means that the matrix representing $A_2 \cdot (A_1 \cdot \mathbf{x})$, which is the matrix representing $T_2 \circ T_1$, is the matrix having for columns the vectors $A_2(\mathbf{c}_1)$, $A_2(\mathbf{c}_2)$ and $A_2(\mathbf{c}_3)$.

We havae therefore found the way to write the matrix product. $A_2 \cdot A_1$ is defined to be the matrix with columns $A_2(\mathbf{c}_1)$, $A_2(\mathbf{c}_2)$ and $A_2(\mathbf{c}_3)$.

Example 4.5. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ be a 2×2 matrix, and let $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$ be a 2×3 matrix. We want to compute the product $A \cdot B$. From the definition found above, in Example 4.4 we have to multiply the left matrix, A, with the columns of the right matrix, B. Then, we have to put the resulting columns in a matrix. So, we have

$$A \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix},$$

$$A \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix},$$

and

$$A \cdot \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}.$$

Putting all the columns together in a matrix, we have

$$A \cdot B = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}.$$

We call the matrix having ones only on the diagonal and zero everywhere else $\mathbb{1}_n$. In other words, $\mathbb{1}_n = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$. For $n = 3$, $\mathbb{1}_3$ takes the form $\mathbb{1}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Theorem 4.6. *Let A be an $m \times n$ matrix, and let B and C have sizes for which the products below all make sense. We have*

- $A(BC) = (AB)C$ *(associative law).*
- $A(B + C) = AB + AC$ *(left distributive law).*
- $(A + B)C = AC + BC$ *(right distributive law).*
- $r(AB) = (rA)B = A(rB)$ *for any number r.*
- $\mathbb{1}_m A = A \mathbb{1}_n = A$.

Remark 4.7. Observe that the commutative law for matrices does not hold in general. Indeed, it is easy to find examples for which $AB \neq BA$.

4.2. **Transpose.** Given an $m \times n$ matrix A, we define the transpose of A, and indicate it by A^T as the $n \times m$ matrix whose rows and columns are swapped.

Example 4.8. Let $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \\ 3 & -2 \end{bmatrix}$, then $A^T = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -1 & -2 \end{bmatrix}$.

So, the first row of A^T is the first column of A and so on.

Theorem 4.9. *If A and B are matrices for which the products and sums below make sense, then we have the equalities*

- $(A^T)^T = A$.

- $(A+B)^T = A^T + B^T$.
- $(rA)^T = rA^T$ *for any number* r.
- $(AB)^T = B^T A^T$.

4.3. **Inverse of a Matrix.** We now consider the problem of finding a multiplicative inverse to matrices. Unlike numbers, matrices do not always have an inverse. We would like to understand when such an inverse exists, how to compute it algorithmically, and what applications inverses can have.

First of all, we have a definition.

Definition 4.10. We say that an $n \times n$ matrix A is *invertible* if there exists a matrix C such that

$$AC = CA = \mathbb{1}_n.$$

In this situation, we say that C is an *inverse* of A. We also sat that an invertible matrix is *nonsingular*, while a matrix that does not have an inverse is said to be *singular*.

Remark 4.11. It turns out that if A has an inverse C, this is uniquely determined. So, C is the only inverse of A. We denote this inverse matrix by A^{-1}.

A relatively obvious application of invertible matrices is the following.

Theorem 4.12. *Let A be an invertible matrix. Then, the equation*

$$A\mathbf{x} = \mathbf{b},$$

has a unique solution for all $\mathbf{b}$*, and it is given by* $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof. By the definition of A^{-1}, we have that

$$A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = \mathbb{1}_n\mathbf{b} = \mathbf{b},$$

therefore showing that $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution. To show that this is the unique solution, consider a vector $\mathbf{u}$ that solves the equation. Then it holds

$$A\mathbf{u} = \mathbf{b}.$$

Multiplying both sides by A^{-1}, we get $\mathbf{u} = A^{-1}\mathbf{b}$, showing that $\mathbf{u}$ is the solution $\mathbf{x}$ given before. □

Now, we would like to understand how to find inverses. For 2×2 matrices, this is not a difficult problem. In fact, we have the following result.

Theorem 4.13. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a* 2×2 *matrix. Then, A is invertible if and only if* $ad - bc \neq 0$*, in which case*

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example 4.14. We can use these results to quickly solve linear systems. For instance, consider the system

$$\begin{cases} 3x_1 + 4x_2 = 3 \\ 5x_1 + 6x_2 = 7. \end{cases}$$

Then, the inverse of the matrix $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$, and the solution of the system is given by

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

If we can decompose a matrix into a product, and we know the inverses of the components, then we can compute the inverse of the product using the following result.

Theorem 4.15. *The following results hold:*

- $(A^{-1})^{-1} = A$.
- $(AB)^{-1} = B^{-1}A^{-1}$.
- $(A^T)^{-1} = (A^{-1})^T$.

An elementary matrix is a matrix that is obtained by performing a single elementary row operation on the identity matrix $\mathbb{1}_n$.

Example 4.16. The following matrices are examples of elementary matrices:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

In fact, A_1 is obtained by summing -4 times the first row to the last row. A_2 is obtained by swapping the first row and the second row. A_3 is obtained by multiplying the last row by 5. These are all elementary row operations.

Let us now consider another matrix (not elementary) given by

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & -1 & 2 \\ 3 & 0 & -1 \end{bmatrix}.$$

Computing the product $A_1 \cdot A$, we see that this is the matrix

$$A_1 \cdot A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & -1 & 2 \\ 3-12 & 0+4 & -1-0 \end{bmatrix}.$$

In other words, multiplying A by A_1 gave us a new matrix which is obtained from A by adding -4 times the first row of A. So, we have performed an elementary operation by multiplying by A_1!

Elementary row operations have some very useful properties:

- If A is an $m \times n$ matrix, and E is an $m \times m$ elementary operation, the product EA gives the matrix obtain from A by applying the elementary operation needed to obtain E from $\mathbb{1}_m$.
- Any elementary row operation E is invertible, and the inverse is the elementary matrix needed to bring E to the identity matrix.

Example 4.17. We want to find the inverse matrix to $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$. To obtain E, we have to perform the operation of summing -4 times the first row of the identity matrix $\mathbb{1}_3$ to the last row. To undo this operation, we need to sum 4 time the first row of the identity matrix $\mathbb{1}_3$ to the last row. So,

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +4 & 0 & 1 \end{bmatrix}$$

This is a very useful characterization of invertible matrices.

Theorem 4.18. *An $n \times n$ matrix A is invertible if and only if is row equivalent to the identity matrix $\mathbb{1}_n$. Moreover, any sequence of elementary operations that reduces A to $\mathbb{1}_n$ will also transform $\mathbb{1}_n$ into A^{-1}.*

Proof. If A is invertible, then from the fact that $A\mathbf{x} = \mathbf{b}$ has a unique solution for every choice of $\mathbf{b}$, it means that A has a pivot in every row. Since the matrix A is a square matrix, pivots have to lie along the diagonal. So, the reduced echelon form of A is $\mathbb{1}_n$. This means that A is row equivalent to $\mathbb{1}_n$.

Conversely, if A is row equivalent to $\mathbb{1}_n$, we can write all the elementary row performed to obtain an echelon form of A as elementary row operations $E_1, \ldots, E_k$. Then, it follows that

$$(E_k \cdots E_1)A = \mathbb{1}_n,$$

which means that $E_k \cdots E_1$ is an inverse (from the left) to A. But since $E_k \cdots E_1$ is invertible, we also have that (show it!)

$$A(E_k \cdots E_1) = \mathbb{1}_n,$$

which means that A is invertible.

Since $A^{-1} = E_k \cdots E_1$, we have that

$$A^{-1} = E_k \cdots E_1 \mathbb{1}_n,$$

which means that A^{-1} is obtained from the identity matrix by applying the same k elementary operations that reduced A to $\mathbb{1}_n$. □

We can then obtain an algorithm for finding A^{-1}:

- Take the matrix A, and define the augmented matrix $A^+ = [A\ \mathbb{1}_n]$, obtained by placing the matrix $\mathbb{1}_n$ close to A.
- Row reduce A^+.
- If A is invertible, we will obtain a new matrix of the form $A^- = [\mathbb{1}_n\ A^{-1}]$. Otherwise, A is not invertible.

Definition 4.19. A *one-to-one* function $f : X \longrightarrow Y$, also called *bijective*, is a function such that the two following conditions are satisfied:

- There are no pairs of elements x_1 and x_2 in X which are different from each other, and such that $f(x_1) = f(x_2)$.
- Whenever we choose some y in Y, this is the image of some x in X. In other words, for any y in Y, we can find and x in X such that $f(x) = y$.

We have the following characterization of invertible matrices.

Theorem 4.20. *Let A be an $n \times n$ matrix. Then, the following statements are equivalent:*

- *A is invertible.*
- *A is row equivalent to $\mathbb{1}_n$.*
- *A has n pivot positions.*
- *The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
- *The columns of A are linearly independent.*
- *The linear map $T(\mathbf{x}) = A \cdot \mathbf{x}$ is one-to-one (bijective).*
- *The equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution for all choices of $\mathbf{b}$.*
- *A^T is invertible.*

4.3.1. *Partitioned Matrices.* A matrix A is said to be *partitioned*, if we have a subdivision of it into sub-matrices. This is usually indicated by some horizontal and vertical line. Matrices of this type look like

$$A = \left[\begin{array}{c|cc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{array}\right].$$

A partition matrix can be seen as a matrix whose entries are matrices themselves. For instance, A can be seen as obtained by putting together three matrices: $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}, \begin{bmatrix} a_{14} \\ a_{24} \end{bmatrix}$.

In general, we can list the entries of a partitioned matrix by indicating the sub-matrices that constitute the partitioned matrix. For instance, we can write:

$$B = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right],$$

for a partitioned matrix B where A_{11}, A_{12}, A_{21} and A_{22} are matrices themselves.

Multiplication of partitioned matrices can be performed using the same rules as matrix multiplication for usual matrices, row-column rule seen before, but using the product of matrices in each term instead of the usual product between numbers. Of course, the partitions need to agree in order to be able to do this.

Example 4.21. Let $A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right]$ and $A = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right]$. We can write $A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$, and $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$. Then, to compute $A \cdot B$ we can perform the product of matrices A and B thinking of the submatrices A_{ij} and B_k as numbers, and using the matrix products for them as well. We have

$$A \cdot B = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix}$$

where now we have to perform the matrix products (the usual ones) also for $A_{11}B_1$ and so on. This gives us our product matrix.

Let us now consider the problem of finding the inverse of partitioned matrices. In particular, we are interested in matrices that can be partitioned in the following form

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline \mathbb{0} & A_{22} \end{array}\right],$$

where A_{11} is a $p \times p$ matrix, A_{22} is $q \times q$, $\mathbb{0}$ is the zero matrix, and A_{12} has size depending on p and q: it is $p \times q$. The equation for finding an inverse to A is

$$AB = \mathbb{1}_{p+q},$$

where B is some general matrix. We therefore have

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline \mathbb{0} & A_{22} \end{array}\right] \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|c} \mathbb{1}_p & \mathbb{0} \\ \hline \mathbb{0} & \mathbb{1} + q \end{array}\right].$$

We call such a matrix *block upper triangular.*

Assume now that A is invertible. Using our previous rules for multiplying partitioned matrices, we find the equations

$$A_{11}B_{11} + A_{12}B_{21} = \mathbb{1}_p \tag{3}$$

$$A_{11}B_{12} + A_{12}B_{22} = \mathbb{0} \tag{4}$$

$$A_{22}B_{21} = \mathbb{0} \tag{5}$$

$$A_{22}B_{22} = \mathbb{1}_q \tag{6}$$

Applying the same reasoingin as in Theorem 4.18, we find right away from Equation (6) that A_{22} is invertible, with inverse $B_{22} = A_{22}^{-1}$. Then, this implies that $B_{21} = \mathbb{0}$, since we can left-multiply both sides of Equation (6). Therefore, Equation (3) becomes $A_{11}B_{11} = \mathbb{1}_p$. This means that A_{11} is invertible. We can multiply Equation (4) on the left by A_{11}^{-1}, and recalling that $B_{22} = A_{22}^{-1}$, we get $B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}$. We have found therefore the inverse of the block matrix A as

$$A^{-1} = \left[\begin{array}{c|c} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ \hline \mathbb{0} & A_{22}^{-1} \end{array}\right].$$

So, this is again block upper triangular.

4.4. **LU factorization.** The diagonal from top-left to bottom-right will be called the *main diagonal* of the matrix. The diagonal from top-right to bottom left will be called the *main anti-diagonal*. If a matrix has only zeros above the main diagonal, then it is said to be *lower triangular*. If it has all zeros below the main diagonal it is called *upper triangular*.

An LU factorization of a matrix A, is a factorization of A as a product $A = LU$ where L is lower triangular, and U is in echelon form, hence it is upper triangular. When we want to obtain a solution of the matrix equation $A\mathbf{x} = \mathbf{b}$, and $A = LU$ has an LU factorization, then we can solve the equation in two steps. Observe that in order to have $LU\mathbf{x} = \mathbf{b}$, we need to have $L\mathbf{y} = \mathbf{b}$, where $\mathbf{y} = U\mathbf{x}$. So, we can solve the system

$$\begin{cases} U\mathbf{x} = \mathbf{y} \\ L\mathbf{y} = \mathbf{b} \end{cases}$$

and a solution of the pair of equations is a solution of the original $A\mathbf{x} = \mathbf{b}$.

The reason why this is usefeful is the following. Suppose we have several equations of type $A\mathbf{x} = \mathbf{b}_1, \dots, A\mathbf{x} = \mathbf{b}_k$ with a very large k. Then, we can solve the first equation using a row reduction argument, and we can obtain an LU factorization of A using the row reduction process. Now, instead of recomputing the row reduction for every equation (observe that we should do it each time we change the augmented matrix, so whenever we change $\mathbf{b}$), we use the argument above to solve the other equations directly through the LU factorization. This speeds up the process significantly.

We show how to construct such a matrix with an algorithm. This algorithm is not always applicable, and sometimes one needs to use permutation matrices as well (PLU and LUP factorizations). We will consider the cases where the factorization exists without permutations. We have a simple preliminary result, first. This result will be used in the algorithm.

Proposition 4.22. *Let A and B be two lower triangular matrices. Then AB is lower traingular as well. Moreover, If A is invertible, then also A^{-1} is lower triangular. Similar statements hold for upper triangular matrices.*

Suppose that A can be reduced to the echelon form U only using row replacements that add a multiple of one row to another row below it. Then we can perform the following procedure:

- Find elementary matrices $E_1, \ldots, E_p$ such that
$$(E_p \cdots E_1)A = U.$$
Here the matrices E_i are all lower triangular because we just need to perform row replacements to rows below. Hence, the product $E_1, \ldots, E_p$ is lower triangular (by Proposition 4.22)
- Then, we can write $A = (E_p \cdots E_1)^{-1}U$. Using Proposition 4.22, we know that $(E_p \cdots E_1)^{-1}$ is lower triangular, and U is upper triangular because it is an echelon form.

4.5. **Subspaces of $\mathbb{R}^n$.** A subspace of $\mathbb{R}^n$ is a set of vectors that behaves like $\mathbb{R}^n$, in the sense that we can add, subtract, multiply by scalars, in the same way as we do in $\mathbb{R}^n$.

We start by giving a definition.

Definition 4.23. A subspace of $\mathbb{R}^n$ is a nonempty set V of $\mathbb{R}^n$ such that the following two properties are satisfied:

- For any pair of vectors $\mathbf{v}, \mathbf{w}$ in V, $\mathbf{v} + \mathbf{w}$ is in V as well.
- For any vector $\mathbf{v}$ in V and any scalar α, the vector $\alpha\mathbf{v}$ is in V as well.

Remark 4.24. Observe that any subspace must contain the zero vector. In fact, since V is nonempty, there exists a vector $\mathbf{v}$ in V. But then $\mathbf{0} = 0 \cdot \mathbf{v}$ is in V for the second property. Or similarly, $\mathbf{0} = \mathbf{v} - \mathbf{v}$ is in V for the first (and second!) property.

Remark 4.25. There are two subspaces of a vector space that are always present. The first one is the zero subspac, consisting of only zero: $\{\mathbf{0}\}$. The other one is $\mathbb{R}^n$ itself. These are called the *trivial* subspaces.

There are two very important subspaces of $\mathbb{R}^n$ that are associated to any matrix (and therefore to any linear map as well!).

Definition 4.26. Let A be matrix. We define the *range* of A (also called the *column subspace*) as the span of the column vectors of A, or equivalently as the span of $A(\mathbf{e}_1), \ldots, A(\mathbf{e}_k)$ where k is the size of the target $\mathbb{R}^k$ of the linear map associated to A. This space is indicated as $\mathrm{Col}(A)$, or $\mathrm{Range}(A)$.

Definition 4.27. The kernel of a matrix A (and threfore of the linear map associated to A) is defined as the subspace of vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. It will be indicated as $\mathrm{Ker}(A)$.

We leave the proof of the next proposition to the reader.

Proposition 4.28. *The kernel of a matrix A is a subspace.*

When describing a subspace, it turns out that we do not need to give a description of all of the vectors, but we can refer to only few vectors that generate the whole subspace as their linear span. However, we would like to have a minimal number of such vectors. This motivation drives the following definition.

Definition 4.29. Let V be a subspace of $\mathbb{R}^n$. A *basis* for V is spanning set of V consisting of linearly independent vectors.

In fact, we have already seen a very important example of basis. Take the trivial subspace $V = \mathbb{R}^n$ of $\mathbb{R}^n$. Then, the canonical vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are a basis for V.

Example 4.30. Consider the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

We want to find a basis for the kernel of A. We need to solve the equation $A\mathbf{x} = \mathbf{0}$ and write a solution in parametric form.

We row reduce the matrix to obtain an echelon form. We find the echelon form

$$B = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This gives us a solution of type $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$ where x_2, x_4 and x_5 are free variables. So, the general vector that solves the equation has the form

$$\begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix},$$

which means that we can write any solution as $\mathbf{x} = x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$, where

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors $\mathbf{u}.\mathbf{v}, \mathbf{w}$ are linearly independent (why?) and therefore they are a basis for the kernel of A.

We now introduce and discuss the notion of coordinate system for a subspace. First, observe that if $\mathcal{B}$ is a basis for the subspace V, then any vector can be written uniquely as a linear combination of vectors of $\mathcal{B}$. In fact, by definition of the fact that $\mathcal{B}$ is a basis, it spans V, meaning that any vector in V is a linear combination of vectors in $\mathcal{B}$. Now, we want to show that this way of writing the elements of $\mathcal{B}$ is unique. Let $v_1, \dots, v_p$ be the vectors of $\mathcal{B}$. Suppose that the vector can be written as a linear combination of $_1, \dots, _p$ in using the coefficients $a_1, \dots, a_p$ and $b_1, \dots, b_p$. In other words, we have that

$$\begin{aligned} \mathbf{x} &= a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p \\ \mathbf{x} &= b_1\mathbf{v}_1 + \cdots + b_p\mathbf{v}_p. \end{aligned}$$

Subtracting the two equations we obtain

$$\mathbf{0} = (a_1 - b_1)\mathbf{v}_1 + \cdots + (a_p - b_p)\mathbf{v}_p.$$

From the definition of basis, we know that $\mathbf{v}_1, \dots, \mathbf{v}_p$ are linearly independent. But the only way that a linear combination of linearly independent vectors could give the zero vector is if the coefficients are all zero. Therefore, we have found that $a_1 = b_1, \dots a_p = b_p$. This shows that there is only one way to write any vector in V is a linear combination of vectors in $\mathcal{B}$.

Given a vector $\mathbb{v}$, we find unique coefficients $a_1, \ldots, a_p$ needed to write $\mathbb{v}$ as a linear combination of vectors in $\mathcal{B}$. We call these coefficients the *coordinates of* $\mathbf{x}$ *with respect to* $\mathcal{B}$, or just the $\mathcal{B}$-coordinates of $\mathbf{x}$. In such a situation we write

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}.$$

It turns out that the number of vectors present in a basis for a subspace V depends only on V, and it is the same between any pair of bases. The following definition is therefore well posed.

Definition 4.31. The dimension of a nonzero subspace V, denoted by dim V, is the number of vectors in any basis for V. We also define the dimension of the zero subspace to be 0.

We define two important dimensions. Namely, the dimensions of the kernel and the range of a matrix.

Definition 4.32. Let A be a matrix. The *rank* of A, written $\text{rank}(A)$ or also $\rho(A)$, is the dimension of the range of A. In other words, we have $\text{rank}(A) = \dim \text{Range}(A)$. The *nullity* of A, written $\text{N}(A)$, is the dimension of its kernel. In other words, $\text{N}(A) = \dim \ker(A)$.

We are now in the position to state the following theorem, proof of which will be given later in the course.

Theorem 4.33. *Let A be a matrix with n columns. Then* $\text{rank}(A) + \text{N}(A) = n$.

Theorem 4.34. *The following statements are equivalent.*

- *The columns of A form a basis or $\mathbb{R}^n$.*
- $\text{Range}(A) = \mathbb{R}^n$.
- $\text{rank}(A) = n$.
- $\text{N}(A) = 0$.
- $\ker(A) = \{\mathbf{0}\}$.

Proof. The proof is simple, and it is left to the reader as an exercise. □

5. Determinants

Determinants have two important types of applications. Firstly, we will see that an $n \times n$ matrix is invertible if and only if its determinant is nonzero. Therefore, determinants characterize whether you can invert or not a matrix. Secondly, determinants have a geometric notion attached to them, where they tell you how much the areas/volumes are modified by linear maps. We first need to define determinants, though.

We have seen before that a 2×2 matrix $A = [a_{ij}]$ is invertible if and only if the quantity $\text{det}A = a_{11}a22 - a_{12}a_{21} \neq 0$. We now want to extend this result to larger square matrices. Let us look at a 3×3 example, before giving the general rule.

Consider the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

and assume that this is an invertible matrix.

Let us multiply second and third rows by a_{11}, if $a_{11} \neq 0$, to obtain the row equivalent matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix}$$

and subtracting multiples of the first row from the second and third rows we find the row equivalent matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a21 & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}.$$

Since A is assumed to be invertible, then we have that in the last matrix either the $(2, 2)$ entry is nonzero, or the $(3, 2)$ entry is nonzero (why?). Assume that the entry $(2, 2)$ is nonzero. Otherwise we can simply exchenge rows and and have again a matrix with $(2, 2)$ entry that is nonzero. Now we multiply the third row by $a_{11}a_{22} - a_{12}a21$, and subtract from it the second row multiplied by $a_{11}a_{32} - a_{12}a_{31}$. We get now a new matrix that is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a21 & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix},$$

where Δ is given by the equation

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

The fact that A is invertible means that we necessarily have $\Delta \neq 0$, or otherwise the last row of the matrix row equivalent to A would be zero, and this is clearly not an invertible matrix.

To interpret this value, Δ, in relation to $\det A$ for 2×2 matrices, observe that we can write Δ in the following way:

$$\Delta = a_{11}\det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

For $n = 1$ we can also set $\det A = a_{11}$, which is the only element in the matrix. With this definition it is easy to see that if A is a 2×2 matrix, $\det A = a_{11} \det[a_{22}] - a_{12} \det[a_{21}]$.

This motivates the following inductive definition of the determinant.

Definition 5.1. For any $n \geq 2$, and any $n \times n$ matrix A, we define the determinant of $A = [a_{ij}]$ in the following way:

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{n+1}a_{1n}A_{1n} = \sum_{j=1}^{n} (-1)^{j+1} a_{1j}A_{1j},$$

where the matrix A_{1j} is obtained from A by deleting the first row, and the j column. In other words, to define the determinant at level n, we use the determinant at level $n - 1$.

Remark 5.2. Observe that the definition is well posed, because we have a starting point, i.e. $n = 1$, and whenever we want to compute the determinant at higher orders, we can reduce it step by step to a determinant of lower order until we reach $n = 1$ (or $n = 2, 3$ since we now have formulas for them as well).

We now define the cofactors of a matrix $A = [a_{ij}]$. The (i, j)-cofactor of A is the number C_{ij} obtained as $C_{ij} = (-1)^{i+j} \det A_{ij}$, where A_{ij} was defined above and it is the matrix obtained from A by removing the i^{th}-row and j^{th}-column.

Using the notion of cofactor, the definition of determinant can be rewritten as

$$\det A = a_{11}C_{11} + \cdots + A_{1n}C_{1n}.$$

In fact, it turns out that the determinant can be equivalently written using any other row rather than the first one, or even columns instead of rows. We formalize this in the next theorem, proof of which will not be given in this notes.

Theorem 5.3. *Let A be an $n \times n$ matrix. Then we have*

$$\det A = a_{i1}C_{i1} + \cdots + a_{in}C_{in}, \tag{7}$$

$$\det A = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj}, \tag{8}$$

for any choice of i and any choice of j.

Remark 5.4. In other words, when computing the determinant we can use an expansion with respect to any choice of row or column. This is useful to simplify computations.

We now consider the determinant of triangular matrices (upper or lower), and see that they are simpler to compute than for other cases.

Theorem 5.5. *Let A be an upper or lower triangular matrix. Then, the determinant of A is the product of the entries along the main diagonal.*

Proof. We assume that our matrix A is upper triangular, just to fix the procedure. A similar reasoning will work for the case of a lower triangular matrix. Observe that the first column of A consists of only zeros and a single possibly nonzero number in position $(1, 1)$. So, a cofactor expansion of $\det A$ along the first column consists of only a term:

$$\det A = a_{11}C_{11}.$$

Now, $C_{11} = \det A_{11}$, so that we need to compute the determinant of the matrix A_{11} obtained from A by deleting first row and first column. To compute this, we can again use the fact that below the top left entry, which now is a_{22} we have only zeros, so that we can again perform a cofactor expansion with respect to the first column to get $\det A = a_{22}C'_{11}$, where C' is a cofactor for the matrix A_{11}, and the prime symbol is simply emphasizing that the matrix is not the original A. Again, this gives us another determinant to compute. But we can iterate this procedure each time, at each step computing the cofactor expansion with respect to the first column of the new matrix, getting at each step a multiplication of a_{kk}, the term along the diagonal, times a new cofactor. Once we get to $k = n - 1$ we find that there is only a term left, namely a_{nn}, which gives the full determinant as $\det A = a_{11} \cdots a_{nn}$, compliting the proof. ☐

5.1. **Properties of determinants.**

Theorem 5.6. *Let A be a square matrix. Then the following facts hold.*

1. *If a multiple of a row of A is added to any other row, then the matrix B obtained satisfies* $\det A = \det B$.
2. *If B is obtained from A by interchanging two rows, then* $\det A = -\det B$.
3. *If B is obtained from A by multiplying a row by a number k, then* $k \det A = \det B$.

Proof. We proceed by induction. This means that we show that the statements hold for some n (e.g. $n = 2$), and then we show that each time the statements hold for some $n = k$, this implies that it also holds for $n = k + 1$. Therefore, we can get it true for any n.

For $n = 2$ it is a simple direct inspection which we leave to the reader as an exercise. Let us now assume that the result holds for $n = k$ for some $k \geq 2$. We want to prove it now for the next integer $n = k + 1$. So, assume that our matrix A has size $n = k + 1$.

Let us show 1. Since the matrix has more than two rows, there is a row of A that does not take part in the elementary operation applied in 1. Let us say that this row is the row i. Using the cofactor expansion in i we find that

$$\det B = a_{i1}(-1)^{i+1} \det B_{in} + \cdots + a_{in}(-1)^{i+n} \det B_{in}.$$

The matrices B_{ik} have been obtained from A_{ik} by performing the same operation on A_{ik} that we performed on A, i.e. multiplying a row by a scalar and adding it to another row. So, we can now apply the inductive hypothesis, since A_{ik} are smaller matrices than A (one row and one column less), and therefore they are of size $k \times k$. Using the inductive hypothesis we get then that $\det B_{ik} = \det A_{ik}$ for all $k = 1, \ldots, n$. This gives then that

$$\det B = a_{i1}(-1)^{i+1} \det A_{in} + \cdots + a_{in}(-1)^{i+n} \det A_{in} = \det A.$$

Let us now show 2. We proceed in the same way to expand the determinant of B with respect to a row that did not take part in the elementary operation of 2. In this case, by the inductive hypothesis, we find that $\det B_{ik} = -\det A_{ik}$, and therefore that

$$\det B = -a_{i1}(-1)^{i+1} \det A_{in} - \cdots - a_{in}(-1)^{i+n} \det A_{in} = -\det A.$$

To show 3, we proceed exactly in the same way, but this time we get multiplying factors of k for each $\det B_{ik}$, and this gives us $\det B = k \det A$. □

The previous results has an incredibly important application, which we now state and prove. Namely, this is the fact that similarly to the 2×2 case seen before, a square matrix has an inverse if and only if its determinant is nonzero.

Theorem 5.7. *Let A be a square matrix. Then A is invertible if and only if* $\det A \neq 0$.

Proof. We know that there exists a matrix U in echelon form that is obtained from A by elementary operations. Then, U is a triangular matrix (by definition of echelon form, and since A is square). From Theorem 5.6 we know that $\det A = (-1)^r \det U$, where r is the number of row interchanges (recall from the theorem that each interchange corresponds to changing sign of the determinant). Then, $\det A \neq 0$ if and only if $\det U \neq 0$. From Theorem 5.5 we know that the determinant of U is just the product of the entries in the diagonal. For a matrix U in echelon form, we also know that being invertible is equivalent to having all nonzero pivots, which all lie in the diagonal since U is square. It follows that U is invertible (has all nonzero pivots) if and only if $\det U \neq 0$. Observing that U is invertible if and only A is invertible now completes the proof, since we have also showed that $\det A \neq 0$ if and only if $\det U \neq 0$. □

The following result is a simple application of the cofactor expansion of determinants with respect to rows and columns. We leave it to the reader as an exercise.

Theorem 5.8. *Let A be a square matrix. Then,* $\det A^T = \det A$.

Lastly, we state and prove another very important property of determinats.

Theorem 5.9. *Let A and B be $n \times n$ matrices. Then,* $\det(AB) = \det A \det B$.

Proof. Suppose first that A is not invertible. Then, by Theorem 5.7 $\det A = 0$, and $\det(AB) = 0$, since AB is not invertible as well, when A is not invertible. Therefore, the equality in the statement of the proof holds true, since it would be $0 = 0$.

Now, let us suppose that A is invertible, and therefore that its determinant is nonzero. From Theorem 4.18 we know that A is row equivalent to $\mathbb{1}_n$ through a sequence of elementary moves $E_1, \dots, E_k$ such that $A = (E_k \cdots E_1)\mathbb{1}_n = E_k \cdots E_1$. Recall that for an elementary operation E, the determinant of E is either 1 or -1, depending on it being as in 1 or 2 of Theorem 5.6, respectively. This means, applying Theorem 5.6 several times, that

$$\begin{aligned}
\det(AB) &= \det(E_k \cdots E_1 B) \\
&= \det(E_k)\det(E_{k-1} \cdots E_1 B) \\
&\ \ \vdots \\
&= \det(E_k) \cdots \det(E_1)\det B \\
&= \det(E_k E_{k-1}) \cdots \det(E_1)\det B \\
&\ \ \vdots \\
&= \det(E_k \cdots E_1)\det B \\
&= \det A \det B,
\end{aligned}$$

which completes the proof. □

5.2. **Cramer's rule.** We already know that if a matrix A has nontrivial determinant, it is possible to find an inverse to it. Now, we would like to find a way to obtain the inverse without the need of algorithmic procedures as we have previously done. This result will be a consequence of the following.

Theorem 5.10 (**Cramer's Rule).** *Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $A\mathbf{x} = \mathbf{b}$ is given by*

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A},$$

for all $i = 1, \dots, n$, where $A_i(\mathbf{b})$ is the matrix obtain from A by replacing the i^{th} column of A by $\mathbf{b}$.

Proof. Denote by $\mathbf{a}_1, \dots, \mathbf{a}_n$ the columns of A, and denote by $\mathbf{e}_1, \dots, \mathbf{e}_n$ the columns of the identity matrix $\mathbf{1}_n$. Suppose that $A\mathbf{x} = \mathbf{b}$, so that we have a solution $\mathbf{x}$ to our matrix equation. Then, using the definition of product multiplication we have

$$\begin{aligned}
A \cdot \mathbb{1}_i(\mathbf{x}) &= A[\mathbf{e}_1 \ \cdots \ \mathbf{x} \ \cdots \mathbf{e}_n] \\
&= [A\mathbf{e}_1 \ \cdots \ A\mathbf{x} \ \cdots A\mathbf{e}_n] \\
&= [\mathbf{a}_1 \ \cdots \ \mathbf{b} \ \cdots \mathbf{a}_n] \\
&= A_i(\mathbf{b}).
\end{aligned}$$

From the multiplicative property of determinants, i.e. Theorem 5.9, we have that

$$(\det A)(\det \mathbb{1}_i(\mathbf{x})) = \det A_i(\mathbf{b}).$$

By direct computation using a cofactor expansion along the i^{th} row, we can see that $\det \mathbb{1}_i(\mathbf{x}) = x_i$. We leave to check this fact to the reader as an exercise. This gives us the result, completing the proof.

□

Let us now define the adjugate matrix. Given an $n \times n$ matrix A, we define the *adjugate* of A, denoted by adj A as the matrix having (i, j)-entry given by

$$C_{ji} = (-1)^{j+i} \det A_{ji},$$

where C_{ji} is the cofactor of A at position (j, i), which we defined as $(-1)^{j+i} \det A_{ji}$.

Now we can derive a formula for the inverse of a matrix.

Theorem 5.11. *Let A be an invertible $n \times n$ matrix. Then, the inverse of A is given by*

$$A^{-1} = \frac{1}{\det A} \text{adj } A.$$

Proof. By definition of inverse, this is a matrix such that $AA^{-1} = \mathbb{1}_n$. Using the definition of multiplication as product of a matrix by the columns of the second matrix, we need to find columns $\mathbf{x}$ for A^{-1} such that $A\mathbf{x} = \mathbf{e}_j$ for all columns j. Using Cramer's rule, this means that the i^{th} entry of $\mathbf{x}$ is given by

$$x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A}.$$

By performing a column expansion along column i of $A_i(\mathbf{e}_j)$ we see that

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji}.$$

This completes the proof. $\square$

5.3. **Areas, Volumes and Determinants.** We show now the relation between determinants and areas and volumes.

Theorem 5.12. *Let A be a 2×2 matrix. Then $|\det A|$ is the area of the parallelogram determined by the columns of A. Similarly, let A be a 3×3 matrix. Then, $|\det A|$ is the volume of the parallelepiped determined by the columns of A.*

Proof. Let us consider the 2×2 case. If the matrix A is diagonal, then the statement is obvious, since the parallelogram is in fact a rectangle, and the area is obviously the product of the two diagonal entries. We want to show now that any 2×2 matrix can be transformed into a diagonal matrix through operations that do not change area and determinant. From Theorem 5.6 we know that performing elementary operations, excluding the rescaling of a row, the absolute value of the determinant is unchanged. Also, since $\det A = \det A^T$, we know that the same holds for columns. In addition, we know that we can obtain a diagonal matrix from A by performing these operations on columns. It just remains to show that also the area of the parallelogram is unchanged when performing such operations. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ denote the columns fo A. We want to show that when changing $\mathbf{a}_2$ to $\mathbf{a}_2 + k\mathbf{a}_1$ (i.e. when we sum a multiple of the first column to the secon column), the area is unchanged. If $\mathbf{a}_2$ is proportional to $\mathbf{a}_1$, the area would be zero, since the parallelogram would collapse to a segment. So, we can assume that $\mathbf{a}_1$ and $\mathbf{a}_2$ are not proportional, which implies that $\mathbf{a}_2 + k\mathbf{a}_1$ is nonzero for any choice of k. Observe that $\mathbf{a}_2$ and $\mathbf{a}_2 + k\mathbf{a}_1$ have both the same perpendicular distance from the line L containing $\mathbf{0}$ and $\mathbf{a}_1$, since they both lie on a line parallel to L. Namely, the line $L + \mathbf{a}_2$, which is just a translation of L. Therefore, the parallelograms so obtained have the same area, since the share the same basis, and they have the same height.

The case of 3×3 matrices is handled in a similar fashion. First, observe that the case when A is diagonal is obviously true. Then, we want to show that we can reduce the general case to the diagonal one. Once again, we need to show that we can add multiples of columns to any other column without changing the volume. The volume of a parallelepiped is determined by the area of the base times the height. Using the same procedure done before, we can change the base into a rectangle without changing the area of the base, hence without changing the volume of the parallelepiped. Now we can perform a translation along a plane without changing the perpendicular projection. This does not change the volume. We leave to the reader the exercise to fill the gaps in this argument, following a reasoning similar to the first part. □

6. Vector Spaces

We now delve into the main topic of Linear Algebra: Vector Spaces. Most of what we have done so far, has implicitly used properties of vector spaces without explicitly saying it. Vector spaces are general objects that abstract the notion of $\mathbb{R}^n$ that we have considered so far. We give the general definition below.

Definition 6.1. A *vector space* is a set V of elements called *vectors* endowed with an operation $+$ which takes two vectors $\mathbf{v}$ and $\mathbf{w}$, and returns a single vector $\mathbf{v}+\mathbf{w}$, and a multiplication by scalars $\cdot$ which takes a number k and a vector $\mathbf{v}$, and returns a vector $k\cdot\mathbf{v}$ (also denoted by kv). These operations satisfy the following defining axioms.

1. $+$ is commutative: $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$.
2. $+$ is associative: $(\mathbf{v}+\mathbf{w})+\mathbf{u}=\mathbf{v}+(\mathbf{w}+\mathbf{u})$.
3. There exists a "zero vector" $\mathbf{0}$ such that $\mathbf{0}+\mathbf{v}=\mathbf{v}$.
4. For any vector $\mathbf{v}$, there exists a vector $-\mathbf{v}$, such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$.
5. The scalar product distributes over $+$: $k(\mathbf{v}+\mathbf{w})=k\mathbf{v}+k\mathbf{w}$.
6. The scalar product distributes over the sum of scalars: $(k_1+k_2)\mathbf{v}=k_1\mathbf{v}+k_2\mathbf{v}$.
7. Scalar product and product among numbers associate: $k_1(k_2\mathbf{v})=(k_1k_2)\mathbf{v}$.
8. $1\mathbf{v}=\mathbf{v}$.

We now give some examples of vector spaces.

Example 6.2. The spaces $\mathbb{R}^n$ that we have considered up to now are all vector spaces, for any choice of n.

Example 6.3. Let V be the set of arrows in the plane or the three dimensional space. Addition here is defined through the parallelogram rule that we have encountered before, and scalar multiplication is obtained by rescaling the size of the arrow. The zero arrow is the zero vector, as one can easily check. This set is a vector space.

Example 6.4. Consider the set $\mathbb{S}$ of sequences of real numbers indexed by the integers $\mathbb{Z}$. These are objects of type

$$\{y_k\}=(\cdots,y_{-2},y_{-1},y_0,y_1,y_2,\cdots),$$

where each y_i is a real number.

We define addition by componentwise addition. This means that given $\mathbf{y}=\{y_k\}$ and $\mathbf{z}=\{z_k\}$ we set

$$\mathbf{y}+\mathbf{z}=\{y_k+z_k\}=(\cdots,y_{-2}+z_{-2},y_{-1}+z_{-1},y_0+z_0,y_1+z_1,y_2+z_2,\cdots).$$

Mnultiplication by scalars is obtained likewise by multiplying all entries of the sequence by the scalar, $c\cdot\{y_k\}=\{cy_k\}$. The zero vector here is the vector consisting of all zeros, $\mathbf{0}=\{y_k\}$ with $y_k=0$ for all k. The reader can verify that this is indeed a vector space.

Example 6.5. Consider the set of polynomials of degree at most n, where $n \geq 0$, and coefficients in the real numbers. An element of this set ihas the general form

$$p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n,$$

where a_i are all real numbers. Then, the sum of two polynomials is simply given by

$$p(x) + q(x) = (a_0 + b_0) + \cdots + (a_n + b_n) x^n,$$

where $p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n$, and $q(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1} + b_n x^n$. Scalar multiplication is given by

$$cp(x) = ca_0 + ca_1 x + \cdots + ca_{n-1} x^{n-1} + ca_n x^n.$$

Example 6.6. Let $F(\mathbb{D})$ be the set of all real valued functions on some subset $\mathbb{D}$ of the real numbers. The usual addition found in calculus, where $f(x) + g(x)$ is componentwise addition, and the multiplication by numbers $kf(x)$, turns this set into a vector space. Here, clearly, the zero vector is just the function $f(x) = 0$ for all $x \in \mathbb{D}$.

Example 6.7. Consider the set of $n \times m$ matrices with real valued entries, $M_{n,m}(\mathbb{R})$. The, the componentwise addition and scalar multiplication that we have defined for matrices turns this space into a vector space. The zero matrix here serves as the zero vector. More generally, linear maps $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ are a vector space with addition and scalar multiplication defined componentwise, and the correspondence that we have studied between matrices and linear maps gives a way of translating one vector space into the other. We will see latern on that there is a very specific name for this situation.

Proposition 6.8. *Let V be a vector space. Then, the following facts hold.*

- $0 \cdot \mathbf{u} = \mathbf{0}$ *for any* $\mathbf{u}$ *in* V*.*
- $k \cdot \mathbf{0} = \mathbf{0}$*, for any number* k*.*
- $-\mathbf{u} = (-1) \cdot \mathbf{u}$ *for any* $\mathbf{u}$ *in* V*.*

Proof. For the first one, we have

$$0 \cdot \mathbf{u} = (0 + 0) \cdot \mathbf{u} = 0 \cdot \mathbf{u} + 0 \cdot \mathbf{u}.$$

Subtracting from both sides $0 \cdot \mathbf{u}$ gives $0 \cdot \mathbf{u} = \mathbf{0}$.

The other two statements can be shown in a similar way, and this is left to the reader as an exercise. ☐

6.1. **Subspaces.** Roughly speaking, a subspace of a vector space V is a subset that is itself a vector space. For instance, if we consider $\mathbf{R}^n$, we can consider only vectors of type $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$, consisting of two arbitrary numbers and always zero in the bottom entry. This is a subspace that greatly resembles $\mathbb{R}^2$. We will later formalize this notion of "resembling".

Definition 6.9. *A subspace W of a vector space V is a subset of V that satisfies the following properties.*

- *The zero vector* $\mathbf{0}$ *is in* W*.*
- W *is closed under addition, meaning that whenever* $\mathbf{v}$ *and* $\mathbf{w}$ *are in* W*, then also* $\mathbf{v} + \mathbf{w}$ *is in* W*.*

- *W is closed under multiplication by scalars. This means that if k is any number and $\mathbf{w}$ is a vector in W, then $k\mathbf{w}$ is in W as well.*

Proposition 6.10. *Let W be a subspace of the vector space V. Then W is a vector space itself.*

Proof. Observe that since the operatrions of $+$ and $\cdot$ are inherited from V, which is a vector space, then they are commutative and associative and distribute over each other. Moreover, since $1 \cdot \mathbf{w} = \mathbf{w}$ for any $\mathbf{w}$ in V, a fortiori this holds when $\mathbf{w}$ is chosen in W since W is a subset of V.

The only things that need to be verified are that $+$ is an operation of W (meaning that is closed under addition), that the scalar product maps into W, and that $\mathbf{0}$ is in W. These are exactly the properties that W needs to verify in order to be called a subspace of V. □

Example 6.11. Given any vector space, there are automatically two subspaces which we will call "trivial" subspaces. Namely, these are the zero subspace, $\{\mathbf{0}\}$ consisting of only the zero vector, and the whole space itself, V.

Example 6.12. Consider the space of polynomials of degree $n > 1$, $\mathbb{P}_n$. For any choice of $0 < m < n$, the space of polynomials of degree m, $\mathbb{P}_m$, is a subspace of $\mathbb{P}_n$ which is not one of the trivial ones trivial.

Example 6.13. In $\mathbb{R}^n$, define W to be the subset consisting of vectors having arbitrary entries everywhere except the bottom entry, which is 0. This is a subspace of $\mathbb{R}^n$.

Given a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in V, we can take linear combinations of them exactly in the same way we did for columns in $\mathbb{R}^n$ until now. We can therefore define the set consisting of all possible linear combinations of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. We call this set the *span* of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, and we write $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, as we did before for column vectors in $\mathbb{R}^n$.

Proposition 6.14. *The span of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in V is a subspace of V.*

Proof. The proof is simple, and left to the reader as an exercise. We only observe that the zero vector $\mathbf{0}$ is in $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ since we know that $0 \cdot \mathbf{u} = \mathbf{0}$ for any vector $\mathbf{u}$. □

Another important class of subspaces is the nullity of a matrix, which is the kernel of a linear map. We have shown this fact. Another important subspace of $\mathbb{R}^n$ is the range of a linear map.

We will now generalize the notion of kernel and range of a linear map to the case of any vector space V. To do so, we have to define linear maps between arbitrary vector spaces.

Definition 6.15. Let V and W be two vector spaces. A function $T : V \longrightarrow W$ is said to be a linear map if it satisfies the following two properties:

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all vectors $\mathbf{u}, \mathbf{v}$ in V.
- $T(k\mathbf{u}) = kT(\mathbf{u})$ for any $\mathbf{u}$ in V and any number k.

Example 6.16. Any matrix A gives rise to a linear map between vector spaces. We have considered this particular situation several times before.

Example 6.17. Let V be the space of differentiable functions defined over the interval $[0, 1]$. Let W be the space of functions defined over $[0, 1]$. Then, the differential operator $\frac{d}{dx}$ from calculus is a linear map $V \longrightarrow W$!

In fact, whenever we have two functions $f(x)$ and $g(x)$ and we sum them, the differential of their sum is simply $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$. Similarly one has $\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x))$.

Then, we can now define the notion of kernel of T for linear maps that deal with arbitrary vector spaces.

Definition 6.18. The *kernel* of the linear map $T : V \longrightarrow W$ is the set of vectors $\mathbf{u}$ in V satisfying the condition that $T(\mathbf{u}) = \mathbf{0}$. We indicate it by the symbol $\ker T$

Similarly, we can define the range of T.

Definition 6.19. Let $T : V \longrightarrow W$ be a linear map. Then, the *range* of T, also called the *image* of T is the set of vectors $\mathbf{w}$ in W such that $T(\mathbf{x}) = \mathbf{w}$ for some $\mathbf{x}$ in V. We indicate it by the symbol ran T.

Definition 6.20. A linear map having zero kernel is said to be *injective*. A linear map having range corresponding to the codomain is said to be *surgective*. If a linear map is both injective and surjective is said to be an *isomorphism*.

Remark 6.21. Isomorphisms are just one-to-one linear maps!

The importance of isomorphisms is that they translate a vector space into another one precisely, without missing anyting or removing anything. Basically, if two vector spaces are isomorphic (i.e. there is an isomorphism between them), it means that they are "substantially" the same as long as linear algebra is involved. We will see this fact in more detail shortly.

6.2. **Linearly Independent Sets and Bases.** Similarly to what we have done with $\mathbb{R}^n$, which is our prototype vector space, we will define linearly independent sets and bases.

Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a set of vectors in V. Then, a *linear combination* of them is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k,$$

where $\alpha_1, \dots, \alpha_k$ are numbers. We will denote the set of linear combinations of $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ as $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$.

We say that a set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent if the only trivial linear combination that it has is obtained through the coefficients $\alpha_1 = \cdots = \alpha_k = 0$. We say that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly dependent if it is not linearly independent.

Theorem 6.22. *If $\mathcal{S} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a set of nonzero vectors, then it is linearly independent if and only if one of the vectors can be written as linear combination of the others.*

Proof. Suppose that one of the vectors is a linear combination of the others. We can assume that $\mathbf{v}_1 = \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k$, or otherwise we can renumber them. Then, we have a nontrivial linear combination of type:

$$\mathbf{v}_1 - \alpha_2 \mathbf{v}_2 - \cdots - \alpha_k \mathbf{v}_k = \mathbf{0}.$$

Viceversa, assume that the vectors are not linearly independent. We can find a combination

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0}.$$

At least one of the α_i needs to be nonzero, by definition of linear dependence. Then, we can write

$$\mathbf{v}_i = -\frac{\alpha_1}{\alpha_i} \mathbf{v}_1 - \cdots - -\frac{\alpha_k}{\alpha_i} \mathbf{v}_k.$$

□

Definition 6.23. Let V be a vector space. Then, we say that $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis of V if the two following conditions are satisfied.

- $\mathcal{B}$ spans V, i.e. span$\mathcal{B} = V$.
- $\mathcal{B}$ is linearly independent.

If $\mathcal{B}$ spans V we say that it is a *spanning set*, even if it is not linearly independent. The definition of basis also applies to the case of subspaces with obvious modifications.

Theorem 6.24 (Spanning Set Theorem). *Let $\mathcal{S} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a set in V. Let W denote the span of $\mathcal{S}$. Then the following hold true.*

1. *If $\mathbf{v}_i$ is a linear combination of the other vectors in $\mathcal{S}$, then if we remove $\mathbf{v}_i$ from $\mathcal{S}$ the remaining vectors still span W.*
2. *If W is not the zero subspace, then some subset of $\mathcal{S}$ is a basis for W.*

Proof. We first prove 1. Upon possibly reordering the vectors in $\mathcal{S}$, we can assume that it is $\mathbf{v}_k$ to be a linear combination of the other vectors. Say, we have

$$\mathbf{v}_k = a_1\mathbf{v}_1 + \cdots + a_{k-1}\mathbf{v}_{k-1}.$$

Consider a linear combination of the vectors in $\mathcal{S}$. This is a vector of type

$$\mathbf{w} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k.$$

Using the assumption on $\mathbf{v}_k$ we get

$$\mathbf{w} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_k a_1\mathbf{v}_1 + \cdots + \alpha_k a_{k-1}\mathbf{v}_{k-1}.$$

Rearranging terms we get

$$\mathbf{w} = (\alpha_1 + \alpha_k a_1)\mathbf{v}_1 + \cdots + (\alpha_{k-1} + \alpha_k a_{k-1})\mathbf{v}_{k-1},$$

which is a linear combination of the vectors in $\mathcal{S}$ excluding $\mathbf{v}_k$. This complets the proof of 1.

To prove 2, if the vectors in $\mathcal{S}$ are linearly independent, then we are done, since we have a basis. If not, at least one of them is a linear combination of the other ones and it can be removed from $\mathcal{S}$ without changing the span. We can keep doing this until we have removed all vectors that are linear combinations of the others, and therefore this is a linearly independent set. Note that this cannot be the set containing only zero vectors since W is not the zero vector subspace. □

6.3. **Coordinate systems.** We will see that vector spaces that admit a finite basis, behave very much like $\mathbb{R}^n$. This notion is formalized through coordinate systems. Roughly speaking, whenever we have a basis consisting of n elements, we can make each of the basis vectors with one of the canonical vectors of $\mathbb{R}^n$, therefore transforming the initial vector space into $\mathbb{R}^n$.

We start by showing an important property of bases.

Theorem 6.25. *Let V be a vector space, and let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ a basis for V. Then, each vector $\mathbf{v}$ in V can be* **uniquely** *decomposed in a linear combination of elements of $\mathcal{B}$:*

$$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n.$$

Proof. The fact that we can decompose any $\mathbf{v}$ in terms of elements of $\mathcal{B}$ is obvious, since a basis is a spanning set for V. The crucial thing to prove here is that this way of writing $\mathbf{v}$ in terms of the elements of $\mathcal{B}$ is unique. Suppose now that we can write $\mathbf{v}$ in two different ways, i.e. we have

$$\begin{aligned} \mathbf{v} &= a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \\ \mathbf{v} &= b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n. \end{aligned}$$

Then, subtracting both equations we get

$$\mathbf{0} = (a_1 - b_1)\mathbf{v}_1 + \cdots + (a_n - b_n)\mathbf{v}_n.$$

Due to the fact that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, it follows that the previous linear combination has to have all trivial coefficients, meaning that $a_1 - b_1 = 0$, and so on up to $a_n - b_n = 0$. This means that all the a_i and the b_i are equal to each others, showing that the way we can write $\mathbf{v}$ in terms of elements of $\mathcal{B}$ is unique. □

The previous result allows us to pose the following definition.

Definition 6.26. Let V be a vector space and let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for it. Given a vector $\mathbf{v}$ in V, we say that the unique numbers $a_1, \ldots, a_n$ that satisfy $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ are the coordinates of $\mathbf{v}$ with respect to $\mathcal{B}$. We indicate them by $[\mathbf{v}]_{\mathcal{B}}$, or simply $[\mathbf{v}]$ for short, when it is clear what basis $\mathcal{B}$ we are using.

Remark 6.27. Observe that when we consider a the coordinates of $\mathbf{v}$, we have n numbers. Putting them in a column, we obtain an element of $\mathbb{R}^n$!

In particular, we can apply this construction to any basis of $\mathbb{R}^n$. in other words, taken a basis $\mathcal{B}$, we can decompose any vector given in terms of the canonical vectors in $\mathcal{B}$. The following example shows this.

Example 6.28. Consider the basis $\mathcal{B}$ consisting of the vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We want to decompose the vector $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in $\mathcal{B}$. So, in other words, we need to find the solutions to the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v},$$

which is

$$x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This is the same as the system of equations (in matrix form)

$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Solving the system using any of the methods that we have studied so far, we find that $x_1 = \frac{2}{3}$ and $x_2 = -\frac{1}{3}$.

Observe that in the previous example, we found a matrix $P_{\mathcal{B}}$ that transformed the coordinates of $\mathbf{v}$ from the basis $\mathcal{B}$ into the canonical basis of $\mathbb{R}^2$. This was obtained by putting the vectors of $\mathcal{B}$ into columns. The procedure in fact is general, and given a basis $\mathcal{B}$ of $\mathbb{R}^n$, if we construct the matrix $P_{\mathcal{B}}$ by placing the vectors of $\mathcal{B}$ in the columns, we find a transformation that takes a vector in the basis $\mathcal{B}$ and returns the decomposition in the canonical basis of $\mathbb{R}^n$.

We will call $P_{\mathcal{B}}$ the change of coordinate matrix. Of course, since $P_{\mathcal{B}}$ is invertible (the columns are linearly independent by construction!), we see that the matrix $P_{\mathcal{B}}^{-1}$ gives us a transformation from the canonical basis to the basis $\mathcal{B}$.

Theorem 6.29. *The mapping corresponding to $P_{\mathcal{B}}$ is one-to-one, i.e. it is an isomorphism.*

Proof. We have already observed that the matrix is invertible. It therefore follows that the linear map corresponding to the matrix is invertible. This means that it is one-to-one. □

Theorem 6.30. *Let V, W be a vector spaces and let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. If we define n vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n$ arbitrarily chosen in W, then there is a unique way of extending this choice to a linear map $T : V \longrightarrow W$, with the property that $T(\mathbf{v}_1) = \mathbf{w}_1, \ldots, T(\mathbf{v}_n) = \mathbf{w}_n$.*

Proof. First, we set $T(\mathbf{v}_i) = \mathbf{w}_i$ for all $i = 1, \ldots, n$. We need to extend this to a linear map over the whole space V. Since $\mathcal{B}$ is a basis, any vector $\mathbf{v}$ admits a unique decomposition in terms of $\mathcal{B}$:

$$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n.$$

Then, we define $T(\mathbf{v}) = a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n$. This definition is well posed, due to the uniqueness of the coefficients a_i. Moreover, the assignment is clearly linear (verify this!). Since $T(\mathbf{v}_i) = \mathbf{w}_i$ by construction, the results is proved. □

6.4. **Dimension.** We now introduce the concept of dimension of a vector space. This is an extremely important notion throughout mathematics and the sciences.

Lemma 6.31. *Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V. Then, any set S consisting of more than n vectors is linearly dependent.*

Proof. Let $p > n$ and consider the set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$. We want to show that S is linearly dependent. We can transform the vectors in S into $\mathbb{R}^n$ using the mapping $P_{\mathcal{B}}$. When we consider the coordinate vectors in $\mathbb{R}^n$ corresponding to the vectors $\mathbf{u}_i$, we obtain p vectors in $\mathbb{R}^n$. They are linearly dependent, since we know that any set with more than n vectors in $\mathbb{R}^n$ is linearly dependent. We can therefore find a nontrivial linear combination

$$c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_p[\mathbf{u}_p]_{\mathcal{B}} = \mathbf{0}.$$

By linearity of the construction of obtaining the coordinate vectors, we have

$$[c_1\mathbf{u}_1 + \cdots c_p\mathbf{u}_p]_{\mathcal{B}} = \mathbf{0}.$$

This means that $P_{\mathcal{B}}(c_1\mathbf{u}_1 + \cdots c_p\mathbf{u}_p) = \mathbf{0}$. But since $P_{\mathbf{B}}$ is an isomorphism, we need to have $c_1\mathbf{u}_1 + \cdots c_p\mathbf{u}_p = \mathbf{0}$, which gives us a nontrivial linear combination of the vectors $\mathbf{u}_i$ that is equal to zero. Therefore, S is linearly dependent. This completes the proof. □

We can now prove the following fundamental result.

Theorem 6.32. *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two bases of V, then they contain the same number of vectors.*

Proof. Let us set n to be the number of vectors of $\mathcal{B}_1$, and m the number of vectors of $\mathcal{B}_2$. Using Lemma 6.31, since $\mathcal{B}_1$ is a basis, $\mathcal{B}_2$ must have at most n vectors, or otherwise it would be linearly dependent. So, we find that $m \leq n$. Now, using the same reasoning but exchanging the roles of $\mathcal{B}_1$ and $\mathcal{B}_2$, we find that $n \leq m$. It follows that $n = m$ and the proof is complete. □

From Theorem 6.32, it follows that the number of vectors found in a basis of a vector space is a well defined quantity that only depends on the vector space.

Definition 6.33. If a vector space V can be spanned by a finite number of vectors we say that it is *finite dimensional.* The number of vectors found in any of its bases is called *dimension* of V, and written $\dim V$. We say that the zero space is zero dimensional. If a space cannot be spanned by any finite number of vectors, then we say that it is *infinite dimensional.*

Let us now consider the case where we have a finite dimensional space V, and a subspace W of V.

Theorem 6.34. *Let W be a subspace of the finite dimensional space V. Any linearly independent set in W can be expanded to a basis of V. Moreover,*

$$\dim W \leq \dim V.$$

Proof. The fact that the dimension of W can at most be the dimension of V is clear, using Lemma 6.31. We just need to show that we can add vectors to any set of linearly independent vectors in W until we obtain a basis of V.

Suppose we have k vectors in W that are linerly independent. Let us call them $\mathbf{v}_1, \dots, \mathbf{v}_k$. If any vector of V is a linear combination of the vectors $\mathbf{v}_i$, then this is a basis. If not, we can find a vector $\mathbf{u}_{k+1}$ that is not a linear combination of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$. These $k+1$ vectors have to be linearly independent, or otherwise $\mathbf{u}_{k+1}$ would be a linear combination of the $\mathbf{v}_i$, against our choice.

We now proceed again with the new vectors $\mathbf{v}_i$ and $\mathbf{u}_{k+1}$. If they span V, we are done. If not, we can find another vector $\mathbf{u}_{k+2}$ that is not in the linear span, and it must therefore be true that all the $\mathbf{v}_i$ and the two $\mathbf{u}_j$ are linearly independent.

Since V is finite dimensional, this process must stop, at which point we have a basis of V. □

Theorem 6.35. *Let V be an n-dimensional vector space. Then, the two following facts hold.*

1. *Any set of n vectors that is linearly independent is a basis.*
2. *Any set ot n vectors that spans V is a basis.*

Proof. We prove 1. If S is a set f n vectors that are linearly independent, we know (Theorem 6.34) that we can extend S to a basis for V. Since V has dimension n, we cannot add any more vectors, and S is already a basis.

We now prove 2. If S spans V, and S has n vectors, we know that a subset of S is a basis of V (because we can prune it down to a set of linearly independent vectors). But since V has dimension n, we cannot remove any vector. So, S is already a basis. □

We now prove the rank theorem.

Theorem 6.36. *Let $T : V \longrightarrow W$ be a linear map, were $\dim V = n$. Then, the following equality holds:*

$$\dim \ker T + \dim \text{rank } T = n.$$

Proof. Take a basis $\mathcal{B}$ for the subspace $\ker T$ of V. Suppose such a basis has k elements, and now extend $\mathcal{B}$ to a basis $\mathcal{P}$ for V. In other words, we add $n-k$ vectors to $\mathcal{B}$. Let us indicate by $\mathbf{v}_i$ with $i = 1, \dots, k$ the elements of $\mathcal{B}$, and by $\mathbb{w}_j$, $j = 1, \dots, n-k$ the elements that we have added to $\mathcal{B}$ to obtain the basis $\mathcal{P}$ of V. The range of T can be written as the image of all vectors in a basis for V. In other words, ran T = span $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_k), T(\mathbf{w}_1), \dots, T(\mathbf{w}_{n-k})\}$. Since $T(\mathbf{v}_i) = \mathbf{0}$ by definition of kernel, it follows that ran T = span $\{T(\mathbf{w}_1), \dots, T(\mathbf{w}_{n-k})\}$. Therefore, a basis of r elements for span $\{T(\mathbf{w}_1), \dots, T(\mathbf{w}_{n-k})\}$ would give that $n = k + r$. We show that $r = n - k$, therefore completing the proof. In other words, we claim that all vectors $T(\mathbf{w}_j)$ are linearly independent. In fact, suppose by way of contradiction that this was not the case. Then, we could find a linear combination (with nontrivial coefficients) such that

$$\alpha_1 T(\mathbf{w}_1) + \cdots + \alpha_{n-k} T(\mathbf{w}_{n-k}) = \mathbf{0}.$$

Using linearity of T we find $T(\alpha_1 \mathbf{w}_1 + \cdots + \alpha_{n-k} \mathbf{w}_{n-k}) = \mathbf{0}$, which means that the vector $\alpha_1 \mathbf{w}_1 + \cdots + \alpha_{n-k} \mathbf{w}_{n-k}$ is in $\ker T$, against the fact that the $\mathbf{w}_j$ were chosen to complement a basis for

$\ker T$, and therefore no linear combination of them could lie in $\ker T$. This contradiction completes the proof. □

7. Spectral Theory

We begin this important component of this course with the following definition of utmost importance.

Definition 7.1. Let $T : V \longrightarrow V$ be a linear map. Then, we say that $\mathbf{v} \neq \mathbf{0}$ is an *eigenvector* of T if $T(\mathbf{v}) = \lambda \mathbf{v}$ for some scalar λ. The scalar λ is said to be an *eigenvalue.* In this situation we say that $\mathbf{v}$ is an *eigenvector associated to* λ.

Remark 7.2. One can reformulate the notions of eigenvector and eigenvalue above in terms of matrices.

In this section we indicate the identity map $\mathbb{1}$, i.e. the linear map that is defined as $\mathbb{1}\mathbf{v} = \mathbf{v}$, exactly as the identity matrix. The following result is relatively obvious, and it is left to the reader as an exercise.

Proposition 7.3. *The set of eigenvectors* $\mathbf{v}$ *associated to the eigenvalue* λ *is the kernel of the map*

$$T - \lambda \mathbb{1},$$

with the exception of $\mathbf{0}$. *Therefore, the set of eigenvectors associated to* λ *with the addition of* $\mathbf{0}$ *is a subspace of* V.

Definition 7.4. The subspace of V which is the kernel of $T - \lambda \mathbb{1}$ is called the *eigenspace* of T associated to λ, and indicated by E_λ.

Let us consider a simple case where we are able to completely determine the eigenvalues of a linear map. We work with a matrix, here.

Theorem 7.5. *Let* A *be an* $n \times n$ *matrix that is in (upper) triangular form. Then the eigenvalues of* A *are exactly the entries on its diagonal.*

Proof. A scalar λ is an eigenvalue if and only if $(A - \lambda \mathbb{1})\mathbf{v} = \mathbf{0}$ for some vector $\mathbf{v} \neq 0$. Let us compute the form of $A - \lambda \mathbb{1}$. We have that

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix}$$

where everything below the main diagonal is zero. Then, $A - \lambda \mathbb{1}$ is given by

$$A - \lambda \mathbb{1} = \begin{bmatrix} a_{11} - \lambda & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} - \lambda \end{bmatrix}$$

where the lambdas appear only along the diagonal, since $\mathbb{1}$ has only ones along the diagonal, and everywhere else it is zero. Then, $(A - \lambda \mathbb{1})\mathbf{v} = \mathbf{0}$ has a nontrivial solution if and only if the equation has a free variable. But this can happen if and only if one of the diagonal entries is zero, due to the fact that below the main diagonal of $A - \lambda \mathbb{1}$ there are only zeros. For a diagonal entry to be zero, we need to have $a_{ii} = \lambda$ for some $i = 1, \ldots, n$. This comples the proof. □

Theorem 7.6. *Let $\mathbf{v}_1, \dots \mathbf{v}_k$ be eigenvectors of T associated to distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Then $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent.*

Proof. Suppose by means of contradiction that the vectors are linearly dependent. The vectors are all nonzero, and therefore one of them must be a linear combination of the preceeding vectors. Let p denote the smallest index such that $\mathbf{v}_p$ can be written as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_{p-1}$. We have

$$c_1\mathbf{v}_1 + \cdots + c_{p-1}\mathbf{v}_{p-1} = \mathbf{v}_p. \tag{9}$$

Applying the linear map T ob both sides of the equation we get

$$c_1\lambda_1\mathbf{v}_1 + \cdots + c_{p-1}\lambda_{p-1}\mathbf{v}_{p-1} = \lambda_p\mathbf{v}_p. \tag{10}$$

Multiplying Equation (9) by λ_p and subtracting Equation (10) from it, we find

$$c_1(\lambda_1 - \lambda_p) + \cdots + \cdots + c_{p-1}(\lambda_{p-1} - \lambda_p)\mathbf{v}_{p-1} = \mathbf{0}.$$

But this means that the vectors $\mathbf{v}_1, \dots, \mathbf{v}_{p-1}$ are linearly dependent, against the fact that p was the smallest index such that this happened. This contradiction shows that all the vectors must be linearly independent. □

7.1. **The Characteristic Equation.** We have the following useful procedure for finding eigenvalues of a matrix.

Theorem 7.7. *Let A be a matrix. Then the eigenvalues λ of A are the roots of the equation* $\det(A - x\mathbb{1}) = 0$.

Proof. To find the eigenvalues of A, we need to find the values λ such that the map $A - \lambda\mathbb{1}$ has nontrivial kernel. This means that the system $(A - \lambda\mathbb{1})\mathbf{v} = \mathbf{0}$ admits nontrivial solutions for $\mathbf{v}$. But this is equivalent to saying that $A - \lambda\mathbb{1}$ is not invertible, which means that $\det(A - \lambda\mathbb{1}) = 0$. So, λ is a root of $\det(A - x\mathbb{1}) = 0$. □

Definition 7.8. The equation $\det(A - x\mathbb{1}) = 0$ whose roots give the eigenvalues of A, is called the *Characteristic Equation*. Observe that the characteristic equation is a polynomial in x (why?), called *characteristic polynomial*.

We now introduce a new notion which is very important in spectral theory. In addition to having theoretical relevance, there also is a practical importance. In fact, several iterative methods for finding eigenvalues of a linearp map are based on this definition.

Definition 7.9. Let A and B be square matrices. Then, we say that they are *similar* if we can find an invertible matrix P such that $B = PAP^{-1}$.

Of course, similarity is a symmetric property, in the sense that if $B = PAP^{-1}$, then we can also find an invertible Q such that $A = QBQ^{-1}$, by just taking $Q = P^{-1}$.

Remark 7.10. If two matrices are similar, it means that they simply represent the same linear map in two different bases.

The following result is very important.

Theorem 7.11. *If A and B are similar, then they have the same characteristic polynomial.*

Proof. Let $B = PAP^{-1}$. Then, we have

$$\begin{aligned}\det(B - x\mathbb{1}) &= \det(PAP^{-1} - x\mathbb{1}) \\ &= \det(PAP^{-1} - Px\mathbb{1}P^{-1}) \\ &= \det[P(A - x\mathbb{1})P^{-1}] \\ &= (\det P) \cdot (\det(A - x\mathbb{1})) \cdot (\det P^{-1}) \\ &= \det(A - x\mathbb{1}),\end{aligned}$$

where we have used the fact that $\det P \cdot \det P^{-1} = \det PP^{-1} = \det \mathbb{1} = 1$. □

By converting a linear map into a matrix, the use of characteristic equations is also applicable to linear maps. In fact, since it does not depend on the basis that we choose to construct a matrix from the linear map, it follows that the approach is well defined.

Definition 7.12. The *algebraic multiplicity* of an eigenvalue is defined to be the multiplicity of λ as a root of the polynomial $\det(A - x\mathbb{1})$. The *geometric multiplicity* of an eigenvalue λ is the dimension of the eigenspace E_λ.

7.2. **Diagonalization.** Diagonalization is a fundamental result in spectral theory, and it is somewhat the motivating result.

Definition 7.13. A square matrix A is said to be *diagonalizable* if it is similar to a diagonal matrix.

The following result is a simplified version of the so-called Spectral Theorem.

Theorem 7.14 (Spectral Theorem)**.** *An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors. Morevoer, $A = PDP^{-1}$ if and only if P consists of columns that are n linearly independent eigenvectors of A, and in this situation D has as diagonal entries the eigenvalues of A.*

Proof. Assume first that A is diagonalizable. Observe that $A = PDP^{-1}$ is equivalent to $AP = PD$, by multiplying the equation on the right by P. Now, suppose that $P = [\mathbf{v}_1 \cdots \mathbf{v}_n]$, where $\mathbf{v}_i$ are the columns of P. Also, assume that D has the values $\lambda_1, \ldots, \lambda_n$ along its diagonal. We have

$$AP = [A\mathbf{v}_1 \cdots A\mathbf{v}_n], \tag{11}$$

and

$$PD = [\lambda_1\mathbf{v}_1 \cdots \lambda_n\mathbf{v}_n]. \tag{12}$$

So, $AP = PD$ means that $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$ for $i = 1, \ldots, n$. Since P is invertible, the columns must be linearly independent, and $\mathbf{v}_i \neq \mathbf{0}$ for all i. Therefore, A has n linearly independent eigenvectors, and these eigenvectors constitute the columns of P. This proves the first part of the if and only if.

Conversely, assume that A has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. We can construct P by using $\mathbf{v}_i$ as columns, and D as the diagonal matrix having λ_i along the diagonal. Using Equation (11) and Equation (12), it follows that $AP = PD$. Since the eigenvectors $\mathbf{v}_i$ are linearly independent, we can find an inverse P^{-1} to P, from which we find $A = PDP^{-1}$. This completes the proof. □

To understand whether a matrix A can be diagonalized, we need to compute the eigenvalues and eigenvectors. If there is a basis of $\mathbb{R}^n$ consisting of eigenvectors, then the matrix is diagonalizable, otherwise it is not. The following result gives us a criterion that is easily applicable to determine if a matrix is diagonalizable, without looking at the eigenvectors.

Theorem 7.15. *An $n \times n$ matrix A that has n distinct eigenvalues can be diagonalized.*

Proof. Since each eigenvector $\mathbf{v}_i$ belongs to a different eigenspace, applying Theorem 7.6 it follows that there are n linearly independent eigenvectors. But n linearly independent eigenvectors in the n-dimensional space $\mathbb{R}^n$ constitute a basis. Using Theorem 7.14 we complete the proof. □

When we have less eigenvalues, we can proceed according to the following result, whose proof we omit.

Theorem 7.16. *Let A be an $n \times n$ matrix with the distinct eigenvalues $\lambda_1, \dots, \lambda_p$.*

- *For each $k = 1, \dots, p$, the geometric multiplicity is at most equal to the algebraic multiplicity.*
- *The matrix A is diagonalizable if and only if the sum of all the geometric multiplicities of the eigenspaces E_k is n. This happens if and only if the characteristic polynomial factors into linear factors, and each geometric multiplicity is the same as the corresponding algebraic multiplicity. Equivalently, this happes if and only if the characteristic polynomial factors into linear factors and the geometric and algebraic multiplicities of each eigenvalue are the same.*
- *If A is diagonalizable and $\mathcal{B}_k$ is the basis for each E_k, then $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ is a basis for $\mathbb{R}^n$.*

Example 7.17. Let us now consider the problem of diagonalizing a matrix. Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

We want to determine whether this matrix is diagonalizable or not, and if the answer is positive, we want to obtain the corresponding diagonal matrix. This means that we want to find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

From Theorem 7.14 we know that A is diagonalizable if and only if we are able to find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A. Alternatively, we can consider Theorem 7.16 and derive the algebraic and geometric multiplicities. This is substantially equivalent.

First of all, we need to compute the eigenvalues, because from that we will be able to compute the eigenvectors. The eigenvalues are the roots of the characteristic polynomial. Therefore, we need to derive the characteristic polynomial and then find its roots. The characteristic polynomial is given by $p(x) = \det(A - x\mathbb{1})$, which is

$$\begin{aligned} p(x) &= \det(A - x\mathbb{1}) \\ &= \det \begin{bmatrix} 1-x & 3 & 3 \\ -3 & -5-x & -3 \\ 3 & 3 & 1-x \end{bmatrix} \\ &= -x^3 - 3x^2 + 4 \\ &= -(x-1)(x+2)^2. \end{aligned}$$

From the factorization of $p(x)$, we find the solutions $x = 1$ and $x = -2$. Therefore, the eigenvalues of A are the numbers $\lambda = 1$ and $\lambda = -2$. Observe that -2 has algebraic multiplicity 2, since it is a root corresponding to a linear factor repeated twice, while 1 has multiplicity 1, being a root of a linear factor with no repetitions. So, we know that if A is diagonalizable, we would need to have two eigenvector corresponding to $\lambda = -2$, and one eigenvector corresponding to $\lambda = 1$. Of course, the eigenvectors corresponding to $\lambda = -2$ and to $\lambda = 1$ are automaatically linearly independent because they correspond to different eigenvalues. The problem is to find two linearly independent vectors that are both eigenvectors of $\lambda = -2$.

Let us first find an eigenvector corresponding to $\lambda = 1$. This is the kernel of the linear map corresponding to the matrix $A - \mathbb{1}$. So, we need to compute

$$\ker \begin{bmatrix} 1-1 & 3 & 3 \\ -3 & -5-1 & -3 \\ 3 & 3 & 1-1 \end{bmatrix}.$$

So, we need to solve the matrix equation

$$\begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving the system gives the only eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Now we have to compute the kernel for $A + 2\mathbb{1}$, and obtain a basis for it. We need to solve the system

$$\det \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We find that the solutions of this equation are vectors $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $x+y+z = 0$. The two vectors $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are a basis of this space, since they are linearly independent, and the space is two dimensional.

We have therefore found three linearly independent eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ for the space $\mathbb{R}^3$. Therefore, applying Theorem 7.14, it follows that A is diagonalizable. Of course, we will have

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

The matrix P is found by placing the vectors $\mathbf{v}_i$, $i = 1, 2, 3$, as its columns. We have

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

This completes the example.

7.3. **Iterative methods for computation of eigenvalues.** We consider now some numerical methods for the computation of eigenvalues. In fact, to determine the eigenvalues of a matrix, we need to solve a polynomial equation. This can be a very hard problem when the degree of the polynomial is higher than 4. In fact, there are no direct formulas that give the roots of a polynomial in general cases for degrees at least 5.

The approach that we are considering now is called "Power Method" or "Power Iteration", because it involves the use of powers of the given matrix. Suppose we have a matrix A, which is $n \times n$. We assume that A has a *strictly dominant eigenvalue*, i.e. an eigenvalue λ_1 such that $|\lambda_1| > |\lambda|$ for all other eigenvalues λ. The method iteratively produces in this case a sequence that converges to the value λ_1, and a sequence of vectors that converges to a corresponding eigenvector.

We assume a simple case where A is diagonalizable, which means that there exists a basis consisting of eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ of $\mathbb{R}^n$. The corresponding eigenvalues are $\lambda_1, \dots, \lambda_n$ (possibly with repetitions), where λ_1 is the dominant eigenvalue whose existence we have assumed in the previous paragraph. We assume here that the eigenvalues are ordered according to

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|.$$

For an arbitrary vector $\mathbf{v} \in \mathbb{R}^n$, then we have

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n.$$

Since A acts on each $\mathbf{v}_i$ as $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$, we have that $A^k\mathbf{v}_i = \lambda_i^k\mathbf{v}_i$. Therefore, we have

$$A^k\mathbf{v} = \alpha_1\lambda_1^k\mathbf{v}_1 + \cdots + \alpha_n\lambda_n^k\mathbf{v}_n,$$

from which we get

$$\frac{1}{\lambda_1^k}A^k\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2(\frac{\lambda_2}{\lambda_1})^k\mathbf{v}_2 + \cdots + \alpha_n(\frac{\lambda_n}{\lambda_1})^k\mathbf{v}_n, \tag{13}$$

Since $|\lambda_1| > |\lambda_i|$ for all $i = 2, \dots, n$, we find that $(\frac{\lambda_i}{\lambda_1})^k \longrightarrow 0$ for all $i \neq 1$ as $k \to \infty$. Therefore, from Equation (13) we obtain that $\frac{1}{\lambda_1^k}A^k\mathbf{v} \longrightarrow \alpha_1\mathbf{v}_1$ as $k \to \infty$. Since $\mathbf{v}_1$ is an eigenvector, the same is true for $\alpha_1\mathbf{v}_1$. Therefore, the procedure converges to an eigenvector of A.

One observation now is that this procedure converges to the "direction" of the egivenvectors, since multiple scalars of eigenvectors are all eigenvectors of the same eigenvalue. But in order to evaluate $\frac{1}{\lambda_1^k}A^k\mathbf{v}$ for large k and obtain $\alpha_1\mathbf{v}_1$, we should also know λ_1, which is what we wanted to know to start with. In general, $A^k\mathbf{v}_1$ will have growing magnitudes in its entries (which are balanced by λ_1^k when you divide by it). So, in practice, at each iteration step, i.e. each time with take the power of A, we rescale the vector dividing by the absolute value of the larges entry of $A^k\mathbf{v}$ so that we bound the size of the entries of $A^k\mathbf{v}$ as k grows. More precisely we will divide by the norm, which is introduced shortly. The algorithm is the following.

Algorithm 1 Power Method for the computation of a dominant eigenvalue.

Require: Matrix A $(n \times n)$ ▷ Given matrix with a dominant eigenvalue λ_1
Ensure: Convergence to λ_1 and to $\mathbf{v}_1$ ▷ Produces λ_1 and a corresponding eigenvector

```
1: Initialize iterations with some vector with largest entry 1: u_0
2: while || u_k/||u_k|| - u_{k+1}/||u_{k+1}|| || > τ do   ▷ Until we do not satisfy some convergence condition
3:     Compute w_{k+1} = A u_k                            ▷ Apply A to previous guess
4:     Compute ||w_{k+1}||
5:     Set u_{k+1} = w_{k+1}/||w_{k+1}||
6: end while
7: Approximate solution of the eigenvalue problem u_k    ▷ Output of the iterative procedure
```

In Algorithm 1, we have used $||\mathbf{v}||$ to indicate the Euclidean distance between $\mathbf{0}$ and $\mathbf{v}$. In other words, if $\mathbf{v} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ we have $\|\mathbf{v}\| = \sqrt{a_1^2 + \cdots + a_n^2}$. This number is called the *norm* of $\mathbf{v}$. When we divide $\mathbf{v}$ by its norm, what we are doing is to determine only the direction of $\mathbf{v}$, and we are discarding the magnitude. In fact, the reader can show as a simple exercise that $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ belongs to the unitary sphere of $\mathbb{R}^n$.

The condition $\|\frac{\mathbf{u}_k}{\|\mathbf{u}_k\|} - \frac{\mathbf{u}_{k+1}}{\|\mathbf{u}_{k+1}\|}\| > \tau$ for convergence, therefore, tells us that until the direction of the iterations does not stabilize we keep performing the iterations. The number τ, called tolerance, is defined before initialization of the iterations, and it determines how accurate our solution will be at the end of the power iterations.

The power method does not converge to any eigenvalue, but rather to the dominant one. However, a modification of the approach can be used to produce other eigenvalues. Suppose in fact that we have a number λ which is close to λ_2. In other words, suppose that we have a guess for the second largest eigenvalue λ_2. Then, taking $\mu_i = \frac{1}{\lambda - \lambda_i}$, for each $i = 1, \ldots, n$, it follows that μ_i is an eigenvalue of $(A - \lambda\mathbb{1})^{-1}$ for each i (check it!) with same eigenvector $\mathbf{v}_i$ as A. Also, in the assumption that λ is closest to λ_2, μ_2 will be a dominant eigenvector, and therefore we can apply the Power Iteration (Algorithm 1) to the matrix $(A - \lambda\mathbb{1})^{-1}$ to obtain the value μ_2, from which we can then obtain λ_2.

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